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EECI – Training - Research - Innovation – European Embedded Control Institute

Dissipativity theory

EPFL- EECI PhD School 2025

Outline

Introduction to dissipativity

- Dissipativity for continuous-time systems
- Induced stability
- Extension to discrete-time systems

Inducing performance

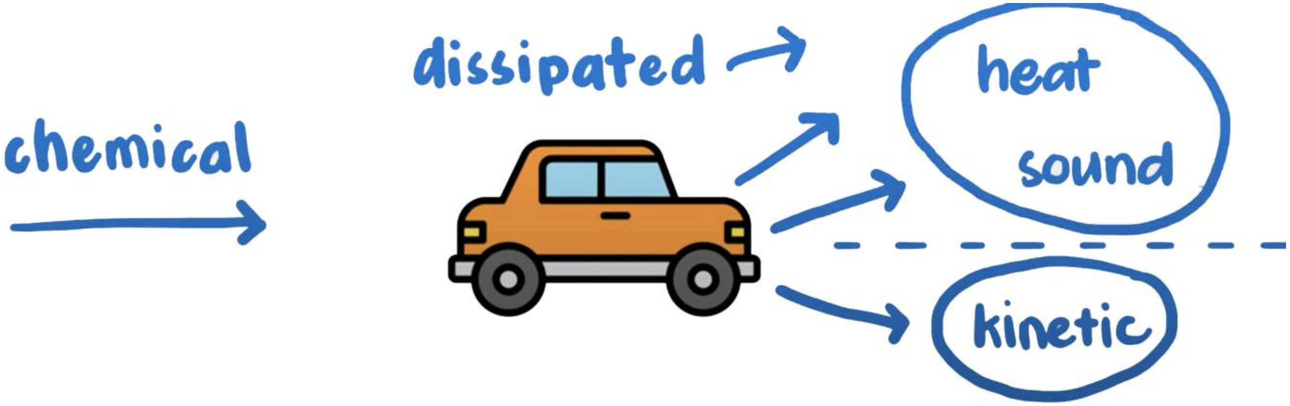
- (Q,S,R) performance
- Performance notions
- Memoryless systems

Dissipativity for feedback interconnections

- Small gain theorem
- Passivity theorem

Incremental properties

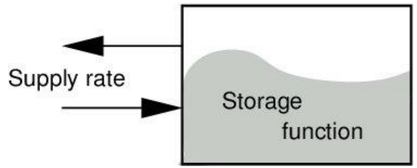
Introduction to dissipativity



Dissipativity theory

Introduced by Jan C. **Willems**[1][2] as a fundamental framework for analyzing the energy balance in dynamical systems.

- generalizes classical notions of passivity and stability by describing systems in terms of an energy-like **storage function** and an input-output relation known as the **supply rate**.



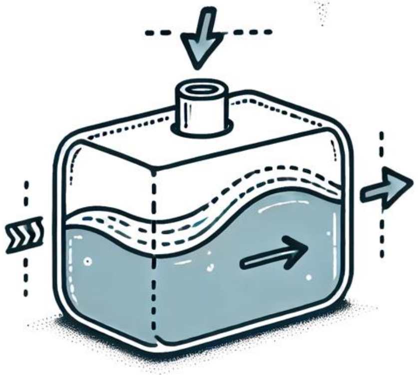
Widely used across various fields in engineering, optimization, and machine learning:

- Control theory and robustness
- Optimization and convergence of algorithms
- Optimal control and Reinforcement Learning
- ***Robustness and stability in neural networks***

[1] J. C. Willems, “Dissipative dynamical systems part i: General theory,” Archive for rational mechanics and analysis, 1972
[2] J. C. Willems, “Dissipative dynamical systems part ii: Linear systems with quadratic supply rates,” Archive for rational mechanics and analysis,1972

Dissipativity for continous-time systems

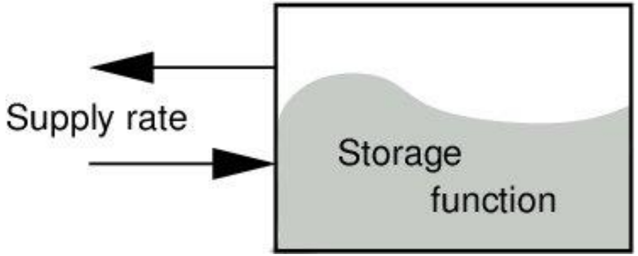
$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ y(t) = h(x(t), u(t)), \end{cases}$$



$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t))dt,$$

Storage and supply

- The **storage function** $V(x)$ can be interpreted as a representation of the stored system's internal energy w.r.t. a single point of neutral storage (energy minimum), where V is zero.
- The **supply rate** $s(u, y)$ quantifies the energy exchanged between the system and its environment.



$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))dt.$$

- $V(x(t_1))$: energy at time t_1
- $V(x(t_0))$: energy possessed at time t_0
- $\int_{t_0}^{t_1} s(u(t), y(t))dt$: amount of energy dissipated

Differentiated dissipation inequality

- With similarities to Lyapunov theory we can also formulate time differentiated version of the classical dissipation inequality.

Differentiated Dissipation Inequality

The nonlinear system (1) is said to be dissipative with respect to a **supply function** $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ if there exists a **storage function** $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) \geq 0$, such that:

$$\frac{d}{dt}V(x(t)) \leq s(u(t), y(t)),$$

for all t and for all $x,\ u,\ y$.

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Induced stability

Lyapunov stability (relaxed form)[*]

The nonlinear system (1) is stable at the origin with input $u(t) = 0$, if there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) \geq 0$ such that:

$$\frac{d}{dt} V(x(t)) \leq 0 \qquad (1)$$

for all t. Moreover, the nonlinear system is asymptotically stable if the condition above holds, but with strict inequality except when $x(t) = 0$.

[*] H. Khalil, Nonlinear systems. 3rd ed. Prentice-Hall, 2002

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for all t. Moreover, the nonlinear system is asymptotically stable if the condition above holds, but with strict inequality except when $x(t) = 0$.

Stability analysis can be performed on a dissipative system by analyzing the supply function.

[*] H. Khalil, Nonlinear systems. 3rd ed. Prentice-Hall, 2002

Induced stability

Stability from dissipativity[*]

Assume the nonlinear system (1) is dissipative under a storage function V w.r.t. a supply rate s that satisfies:

$$s(0, y) \leq 0$$

for all y, then, the nonlinear system is stable. If the supply function satisfies the condition above, but with strict inequality when $y \neq 0$, and the system is observable, then the nonlinear system is asymptotically stable.

- Unlike Lyapunov theory, dissipativity is not restricted to stability
  - **storage function** connects classical dissipativity to **stability**.
  - **supply rate** connects dissipativity to various **performance** notions.

[*] A. Van der Schaft, L2-gain and passivity techniques in nonlinear control. Springer, 2000.

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Dissipativity for discrete-time systems

$$\begin{cases} x(k+1) = f(x(k), u(k)), \\ y(k) = h(x(k), u(k)), \end{cases}$$

$$V(x(k_1+1)) - V(x(k_0)) \leq \sum_{k=k_0}^{k_1} s(u(k), y(k)),$$

Dissipativity for discrete-time systems

- Discrete-time nonlinear dynamical systems

$$\begin{cases} x(k+1) = f(x(k), u(k)), & f(0,0) = 0 \quad (1a) \\ y(k) = h(x(k), u(k)), & h(0,0) = 0 \quad (1b) \end{cases}$$

Dissipativity[*]

The nonlinear system (1) is said to be dissipative with respect to a **supply function** $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ if there exists a **storage function** $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) \geq 0, \forall x$ and:

$$V(x(k_1+1)) - V(x(k_0)) \leq \sum_{k=k_0}^{k_1} s(u(k), y(k)),$$

for all k_0, k_1 with $k_1 \geq k_0$ and for all input u.
Alternatively in difference form, such that

$$V(x(k+1)) - V(x(k)) \leq s(u(k), y(k)),$$

for all k and for all x, u, y.

[*] C. I. Byrnes and W. Lin, "Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems," IEEE Transactions on automatic control, 1994.

Inducing performance

$$\begin{bmatrix} u \\ y \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

(Q,S,R) framework

- The supply function connects dissipativity to various performance notions.
  - A popular choice of supply function is the class of quadratic ones of the form:

$$s(u, y) = \begin{bmatrix} u \\ y \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix},$$

(Q, S, R) real symmetric matrices

- With the (Q,S,R) framework we can consider the following performance notions:
  - Finite $\mathcal{L}_2$ gain
  - Passivity
  - Strictly output passivity

Performance notions

- **Finite $\mathcal{L}_2$ gain**: characterizes the maximum amplification of input energy to output energy. The supply rate is defined as:

$$s(u, y) = \gamma^2 \|u\|^2 - \|y\|^2$$

which corresponds to the (Q, S, R) matrices:

$$Q = \gamma^2 I, \quad S = 0, \quad R = -I.$$

or equivalently

$$Q = \gamma I, \quad S = 0, \quad R = -\gamma^{-1} I.$$

- ➢ stronger statement than stability; it bounds how much the output of a systems can grow with respect to its input.

Performance notions

- **Passivity**: specific case of dissipativity where the system does not generate energy. The supply rate for passivity is:

$$s(u, y) = y^\top u,$$

which corresponds to the (Q, S, R) matrices:

$$Q = 0, \quad S = \frac{1}{2} I, \quad R = 0.$$

Performance notions

- **Strictly Output Passivity**: Strictly output passive systems are those that are passive with an additional output-dependent dissipation term.
The supply rate is:

$$s(u, y) = y^\top u - \epsilon \|y\|^2, \quad \epsilon > 0,$$

which corresponds to the (Q, S, R) matrices:

$$Q = 0, \quad S = \frac{1}{2} I, \quad R = -\epsilon I.$$

or equivalently:

$$Q = 0, \quad S = I, \quad R = -2\epsilon I.$$

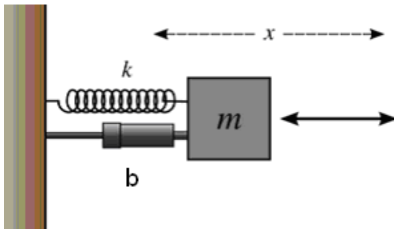
In addition, output strict passivity implies an $\mathcal{L}_2$ gain of $\gamma = 1/\epsilon$ because a completion of squares argument gives

$$u^\top y - \frac{1}{\gamma} y^\top y \leq \frac{\gamma}{2} u^\top u - \frac{1}{2\gamma} y^\top y = \frac{1}{2\gamma}(\gamma^2 |u|^2 - |y|^2).$$

Then the storage function $2\gamma V(\cdot)$ yields the $\mathcal{L}_2$-gain supply rate $\gamma^2 |u|^2 - |y|^2$.

Example mass-spring-damper system

- Equation of motion $m\ddot{x} + b\dot{x} + kx = F$



- Totale energy of the system (kinetic and potential)

$$E_{tot} = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

- Rate of change of the total energy

$$\frac{dE_{tot}}{dt} = F\frac{dx}{dt} - b\left(\frac{dx}{dt}\right)^2$$

By rewriting the terms and considering $y = \dot{x}$ and $u = F$, we obtain that the system is strictly output passive

$$\frac{dE_{tot}}{dt} \leq uy - by^2$$

$$\frac{dE_{tot}}{dt} \leq \begin{bmatrix} u \\ y \end{bmatrix}^\top \begin{bmatrix} 0 & 1/2 \\ 1/2 & -b \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

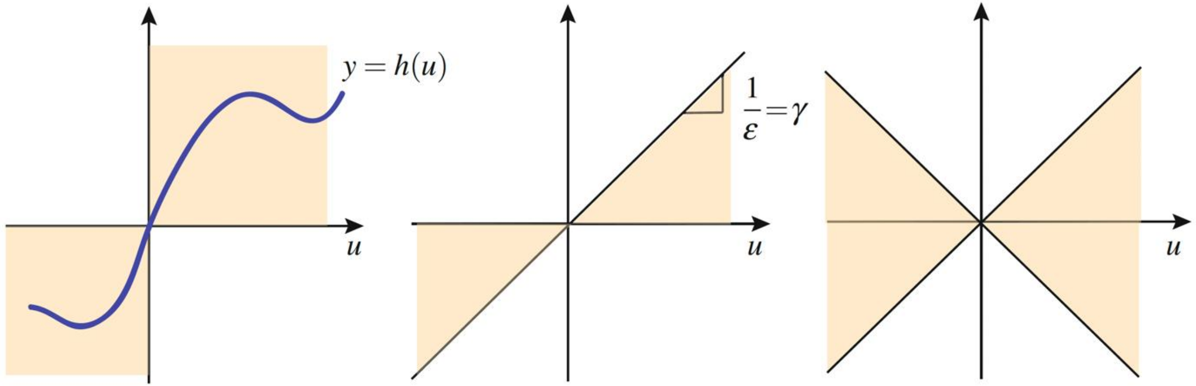
Graphical interpretation

- For a memoryless system

$$y(t) = h(u(t))$$

we take the storage function to be zero and interpret dissipativity as the static inequality

$$s(u, h(u)) \geq 0, \quad \forall u \in \mathbb{R}^m,$$



passivity	output strict passivity	finite gain
$uh(u) \geq 0$	$uy - \epsilon y^2 \geq 0$	$\gamma^2 u^2 - y^2 \geq 0$

Slope restricted nonlinearities

- A function $\sigma(\cdot)$ is said to have a slope restriction $[0, \gamma]$ if

$$0 \leq \frac{\sigma(x) - \sigma(y)}{x - y} \leq \gamma, \quad \forall x, y \in \mathbb{R}, \quad \gamma > 0.$$

Thus, we have

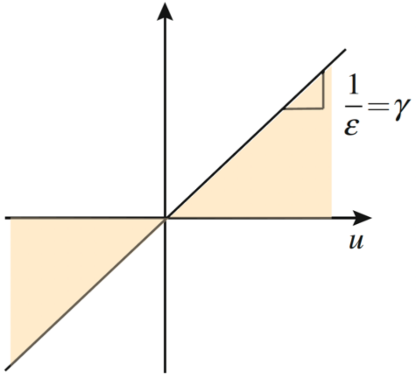
$$\Delta u \gamma \geq \Delta y$$

or equivalently in a quadratic form:

$$\Delta u \Delta y \gamma - \Delta y^2 \geq 0$$

Leading to the following supply

$$s(\Delta u, \Delta y) = \begin{bmatrix} \Delta u \\ \Delta y \end{bmatrix}^\top \begin{bmatrix} 0 & I \\ I & -2\epsilon I \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta y \end{bmatrix}.$$



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LMI to certify finite L2 gain

- Discrete-time linear time-invariant (LTI) system:

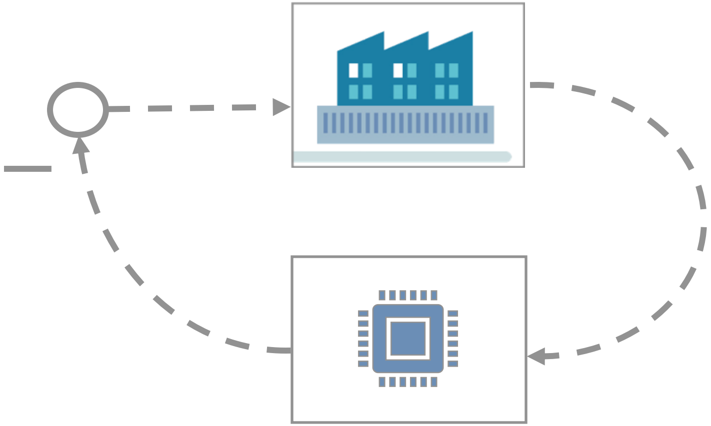
$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k), \\
y(k) &= Cx(k) + Du(k),
\end{aligned}
$$

- We aim to certify the $\mathcal{L}_2$-gain $\gamma > 0$ of the system using:
  - A *quadratic storage function*: $V(x) = x^\top P x$, with $P \succ 0$,
  - A *quadratic supply rate*: $s(u(k), y(k)) = \gamma^2 u(k)^\top u(k) - y(k)^\top y(k)$.
- LMI for certifying the $\mathcal{L}_2$-gain γ (also known as Bounded Real Lemma):

$$
\begin{bmatrix} A^\top P A - P & A^\top P B & C^\top \\ \left(A^\top P B\right)^\top & B^\top P B - \gamma I & D^\top \\ C & D & -\gamma I \end{bmatrix} \preceq 0.
$$

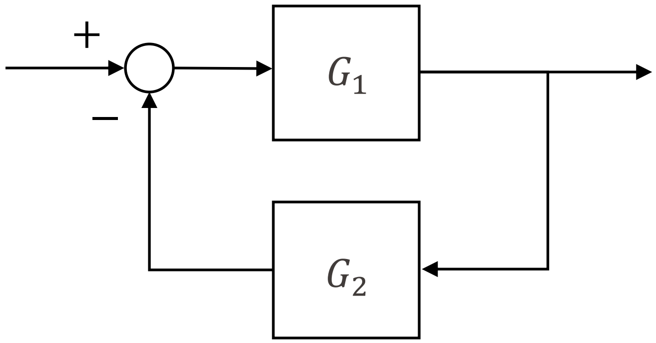
where $P \succ 0$ is the decision variable, and $\gamma > 0$ is the $\mathcal{L}_2$-gain.

Dissipativity for feedback interconnections



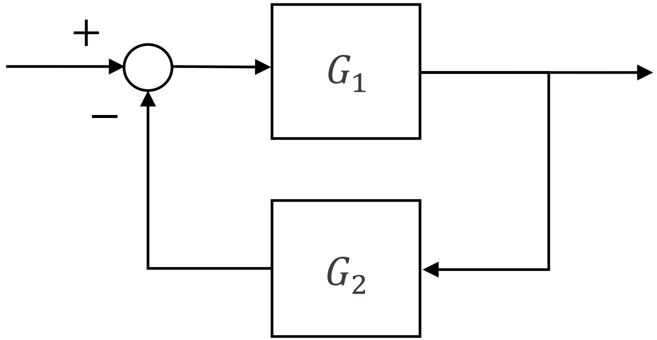
Stability for feedback interconnection

- Dissipativity theory provides a powerful framework for analyzing feedback systems by characterizing system behavior in terms of energy storage and dissipation, enabling the derivation of stability analysis:
  - Small gain theorem.
  - Feedback theorem for passive systems.



Small gain theorem

- Consider two systems with gains γ_1 and γ_2.



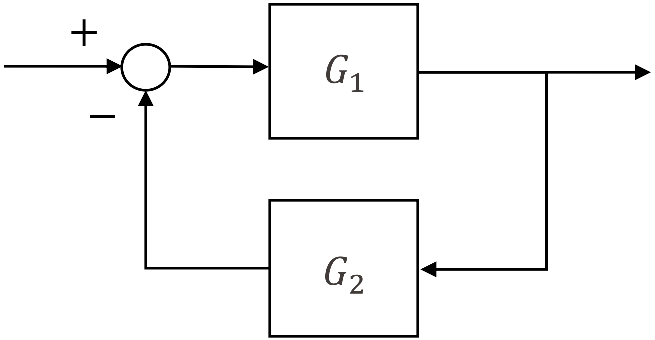
The **small gain theorem** ensures stability when the product of the $\mathcal{L}_2$-gains of two interconnected systems is less than one.
The interconnected system is stable[*] if: $\gamma_1\gamma_2 < 1$.

➢ If the system has a finite $\mathcal{L}_2$ gain, we can design a controller with a finite $\mathcal{L}_2$ gain such that $\gamma_1\gamma_2 < 1$.

[*] Van der Schaft, A. (2000). *L2-gain and passivity techniques in nonlinear control*. Berlin, Heidelberg: Springer Berlin Heidelberg.

Passivity theorem

- Consider two systems G_1 and G_2 interconnected via negative feedback, i.e., $u_1 = -y_2$, $u_2 = y_1$.



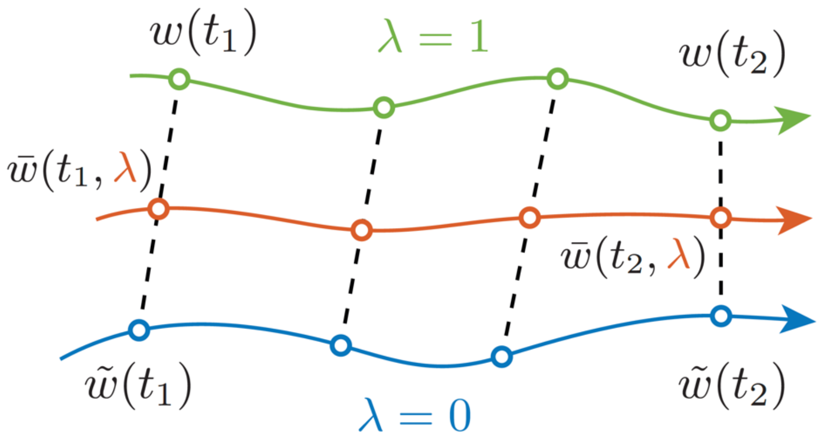
If both systems are strictly passive, the **passivity theorem** tells us that interconnection is passive and stable[*].

➢ If the system is strictly passive it can be desirable to enforce strict passivity of the controller.

[*] Van der Schaft, A. (2000). *L2-gain and passivity techniques in nonlinear control*. Berlin, Heidelberg: Springer Berlin Heidelberg.

[*] Zakwan, M., & Ferrari-Trecate, G. (2024). *Neural Port-Hamiltonian Models for Nonlinear Distributed Control: An Unconstrained Parametrization Approach*. arXiv preprint arXiv:2411.10096.

Incremental dissipativity



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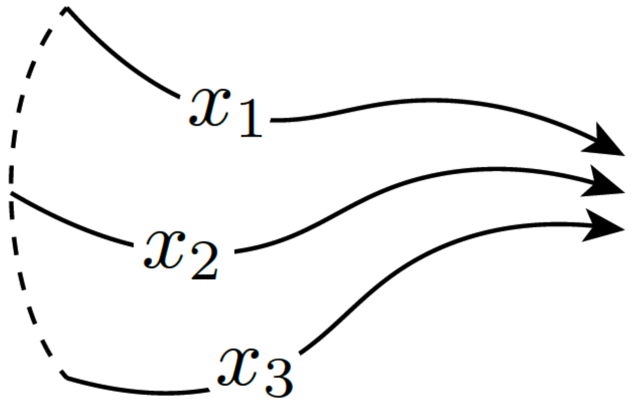
  Incremental dissipativity allows us to compare the system's behavior *across different trajectories.*



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  By analyzing how *any pair* of system trajectories relate energetically, we can derive stronger statements about tracking performance, convergence to desired trajectories, and robustness.

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  Many advanced control problems (e.g., trajectory tracking, adaptive control) benefit from incremental properties because they do not rely on a single operating condition.

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- **Key for Nonlinear & Time-Varying Systems**
  Many advanced control problems (e.g., trajectory tracking, adaptive control) benefit from incremental properties because they do not rely on a single operating condition.
- **Same Tools, New Perspective**
  The underlying idea is similar to classical dissipativity—there is still a storage function and a supply rate. But we extend them to capture differences between pairs of trajectories rather than absolute values.

Incremental dissipativity

Consider a discrete-time nonlinear system:

$$\begin{cases} x(k+1) = f(x(k), u(k)), & \text{(1a)} \\ y(k) = h(x(k), u(k)), & \text{(1b)} \end{cases}$$

Incremental Dissipativity[*]

System (1) is incrementally dissipative if there exists a storage function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that for all time steps k and for any two trajectories (x, u, y) and (x', u', y'):

$$V(x(k+1), x'(k+1)) - V(x(k), x'(k)) \leq s(u(k), u'(k), y(k), y'(k)),$$

where $s(u, u', y, y')$ is the incremental supply rate, quantifying energy flow between different system trajectories.

[*] P. J. Koelewijn and R. T´oth, "Incremental stability and performance analysis of discrete-time nonlinear systems using the lpv framework," IFAC-PapersOnLine, vol. 54, no. 8, pp. 75–82, 2021.

Relation to stability

Incremental Stability[*]

If a system is incrementally dissipative with a storage function V and the supply function satisfies:

$$s(u, u, y, y') < 0, \quad \forall u, \quad \forall y \neq y',$$

then the system exhibits incremental asymptotic stability, meaning that all trajectories asymptotically converge toward one another.

Incremental dissipativity can be analyzed using a quadratic supply rate (Q, S, R) function:

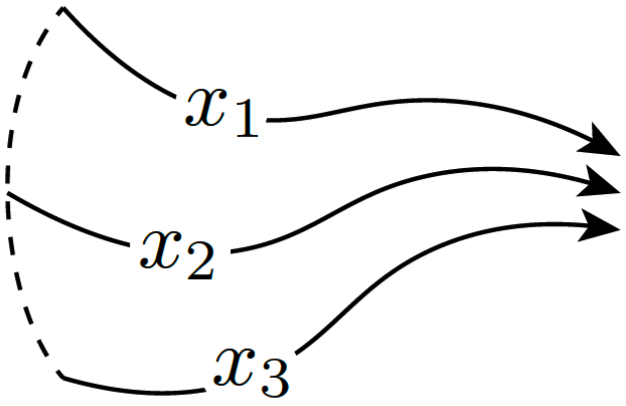
$$s(u, u', y, y') = \begin{bmatrix} u - u' \\ y - y' \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} u - u' \\ y - y' \end{bmatrix}.$$

For appropriate values of Q, S, R, this formulation covers notions such as incremental passivity and incremental $\mathcal{L}_2$-gain.

[*] P. J. Koelewijn and R. Toth, "Incremental stability and performance analysis of discrete-time nonlinear systems using the lpv framework," IFAC-PapersOnLine, vol. 54, no. 8, pp. 75–82, 2021.

Conclusions

- Incremental stability is a ***stronger property*** than standard stability, because if all trajectories converge to each other (the essence of incremental stability), then in particular they converge to (and thus remain stable with respect to) any one trajectory—such as an equilibrium.



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