Some Exercises for Chapter 3 of Advanced Control Systems

Problem 3.1: Consider a PI controller in a unity feedback system as:

$$K(s) = K_P + \frac{K_I}{s}$$

1. Find a discrete representation of the PI controller $K(q^{-1})$. You can use the transformation $s = (1 - q^{-1})/h$ where h is the sampling period.
2. Find an equivalent RST controller for the PI controller.

Solution: Using the backward transformation we obtain:

$$K(q^{-1}) = K_P + \frac{K_I h}{1 - q^{-1}} = \frac{(K_P + K_I h) - K_P q^{-1}}{1 - q^{-1}}$$

Note that $u(k) = K(q^{-1})[r(k) - y(k)]$ or:

$$(1 - q^{-1})u(k) = [(K_P + K_I h) - K_P q^{-1}]r(k) - [(K_P + K_I h) - K_P q^{-1}]y(k)$$

Comparing the above equation with the control law in an RST controller

$$S(q^{-1})u(k) = T(q^{-1})r(k) - R(q^{-1})y(k)$$

leads to:

$$\begin{aligned} R(q^{-1}) &= (K_P + K_I h) - K_P q^{-1} \\ S(q^{-1}) &= 1 - q^{-1} \\ T(q^{-1}) &= (K_P + K_I h) - K_P q^{-1} \end{aligned}$$

Problem 3.2: Consider a control system with a feedback and feedforward controller as shown below:

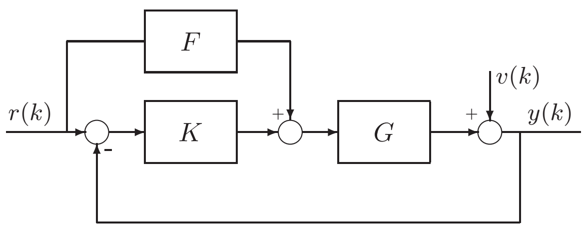

Fig. 1: Two-degree of freedom controller

where $K(q^{-1}) = \frac{N_k(q^{-1})}{D_k(q^{-1})}$ and $F(q^{-1}) = \frac{N_f(q^{-1})}{D_f(q^{-1})}$. Find an equivalent RST controller.

Solution: From the block diagram we can find the control law as follows:

$$u(k) = K(q^{-1})[r(k) - y(k)] + F(q^{-1})r(k)$$

which can be written as:

$$D_f(q^{-1})D_k(q^{-1})u(k) = D_f(q^{-1})N_k(q^{-1})[r(k) - y(k)] + D_k(q^{-1})N_f(q^{-1})r(k)$$

Comparing the above equation with the control law in an RST controller, we obtain:

$$\begin{aligned} R(q^{-1}) &= D_f(q^{-1})N_k(q^{-1}) \\ S(q^{-1}) &= D_f(q^{-1})D_k(q^{-1}) \\ T(q^{-1}) &= D_f(q^{-1})N_k(q^{-1}) + D_k(q^{-1})N_f(q^{-1}) \end{aligned}$$

Problem 3.3: Consider the following plant model:

$$G(z) = \frac{0.5(z - 0.8)}{(z - 0.7)(z - 0.5)}$$

Take $P_d(q^{-1}) = 1 - 0.7q^{-1}$ and the same dynamics for tracking and regulation.

1. Compute an RST controller.
2. Redesign the RST controller such that it includes an integrator.
3. Redesign the RST controller such that it includes an integrator using Q-parametrization (see Example 3.8).

Solution: In the first step $G(z)$ should be converted to $G(q^{-1})$ by replacing z with q:

$$G(q^{-1}) = \frac{0.5q - 0.4}{q^2 - 1.2q + 0.35} = \frac{0.5q^{-1} - 0.4q^{-2}}{1 - 1.2q^{-1} + 0.35q^{-2}}$$

Therefore, $n_A = 2$, $n_B = 2$ and $d = 1$.

1. We have $n_R = n_A - 1 = 1$ and $n_S = n_B - 1 = 1$, therefore, the Bezout equation is:

   $$(1 - 1.2q^{-1} + 0.35q^{-2})(1 + s_1q^{-1}) + (0.5q^{-1} - 0.4q^{-2})(r_0 + r_1q^{-1}) = 1 - 0.7q^{-1}$$

   By making equal the coefficients of the same power of q on both sides, we obtain:

   $$\begin{aligned} -1.2 + s_1 + 0.5r_0 &= -0.7 \\ 0.35 - 1.2s_1 - 0.4r_0 + 0.5r_1 &= 0 \qquad \Rightarrow s_1 = 1.333, r_0 = -1.666, r_1 = 1.1667 \\ 0.35s_1 - 0.4r_1 &= 0 \end{aligned}$$

   Then $T(q^{-1}) = P(1)/B(1) = 0.3/0.1 = 3$.

2. To have an integrator in the controller we should take $H_S(q^{-1}) = 1 - q^{-1}$ and define

   $$A'(q^{-1}) = A(q^{-1})H_S(q^{-1}) = 1 - 2.2q^{-1} + 1.55q^{-2} - 0.35q^{-3}$$

   Then $n_{R'} = 2$ and we should solve:

   $$(1 - 2.2q^{-1} + 1.55q^{-2} - 0.35q^{-3})(1 + s_1q^{-1}) + (0.5q^{-1} - 0.4q^{-2})(r_0 + r_1q^{-1} + r_2q^{-2}) = 1 - 0.7q^{-1}$$

Which leads to the following matrix equality using the Sylvester matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2.2 & 1 & 0.5 & 0 & 0 \\ 1.55 & -2.2 & -0.4 & 0.5 & 0 \\ -0.35 & 1.55 & 0 & -0.4 & 0.5 \\ 0 & -0.35 & 0 & 0 & -0.4 \end{bmatrix} \begin{bmatrix} 1 \\ s_0' \\ r_0 \\ r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.7 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution to this equation is:

$$\begin{aligned} R(q^{-1}) &= 21.667 - 26.833q^{-1} + 8.1667q^{-2} \\ S(q^{-1}) &= 1 - 10.333q^{-1} + 9.333q^{-2} \end{aligned}$$

For an RST controller with integrator the polynomial $T(q^{-1})$ will be $T(q^{-1}) = R(1) = 3$.

3. Instead of a direct method performed in item 2, we can use Q-parameterization as follows. We take the solution of item 1 as $R_0(q^{-1}) = -1.666 + 1.1667q^{-1}$ and $S_0(q^{-1}) = 1 + 1.333q^{-1}$ and $Q(q^{-1}) = q_0$. Then

$$\begin{aligned} R(q^{-1}) &= R_0(q^{-1}) + q_0 A(q^{-1}) \\ S(q^{-1}) &= S_0(q^{-1}) - q_0 B(q^{-1}) \\ S(1) &= 0 \end{aligned}$$

Note that the last equation $S(1) = 0$ guarantees the existence of an integrator in the controller. From this equation we find $2.333 - 0.1q_0 = 0$ which gives $q_0 = 23.333$. Therefore,

$$\begin{aligned} R(q^{-1}) &= -1.666 + 1.1667q^{-1} + 23.333(1 - 1.2q^{-1} + 0.35q^{-2}) = 21.667 - 26.833q^{-1} + 8.1667q^{-2} \\ S(q^{-1}) &= 1 + 1.333q^{-1} - 23.333(0.5q^{-1} - 0.4q^{-2}) = 1 - 10.333q^{-1} + 9.333q^{-2} \end{aligned}$$

Problem 3.4: Consider the following plant model:

$$G(q^{-1}) = \frac{0.5q^{-1} - 0.4q^{-2}}{1 - 1.2q^{-1} + 0.35q^{-2}}$$

For the regulation dynamic choose the dominant closed loop pole to have a damping of $\zeta = 0.8$ and a natural frequency of $\omega_n = 1$ rad/s. Note that $h = 1$ s.

1. Design an RST controller for Model Reference Control problem.
2. Add a fixed term $H_R(q^{-1}) = 1 + q^{-1}$ in the controller.
3. Repeat the second item using Q-parametrization.

Solution: For the MRC problem, we should check first if the zeros of $B^\star(q^{-1})$ are inside the unit circle. We have: $B^\star(q^{-1}) = 0.5 - 0.4q^{-1}$, therefore, $0.5z - 0.4 = 0$ gives $z = 0.8$ which is inside the unit circle. Then, we should compute the closed-loop polynomial $P_d(q^{-1})$. Using (3.37) and (3.38) we obtain:

$$\begin{aligned} p_1 &= -2e^{-\zeta\omega_n h} \cos\left(\omega_n h\sqrt{1-\zeta^2}\right) = -0.7417 \\ p_2 &= e^{-2\zeta\omega_n h} = 0.2019 \end{aligned}$$

Therefore, $P_d(q^{-1}) = 1 - 0.7417q^{-1} + 0.2019q^{-2}$ and we choose $P(q^{-1}) = P_d(q^{-1})B^\star(q^{-1})$.

1. For RST controller design we should solve the following equation:

$$A(q^{-1})S(q^{-1}) + q^{-1}B^\star(q^{-1})R(q^{-1}) = P_d(q^{-1})B^\star(q^{-1})$$

This equation has a solution if $S(q^{-1}) = S'(q^{-1})B^\star(q^{-1})$, that leads to:

$$A(q^{-1})S'(q^{-1}) + q^{-1}R(q^{-1}) = P_d(q^{-1})$$

We have $n_R = n_A - 1 = 1$ and $n_{S'} = 1 - 1 = 0$. Thus $R(q^{-1}) = r_0 + r_1q^{-1}$ and $S'(q^{-1}) = 1$:

$$1 - 1.2q^{-1} + 0.35q^{-2} + r_0q^{-1} + r_1q^{-2} = 1 - 0.7417q^{-1} + 0.2019q^{-2}$$

Solving the equation gives $r_0 = 0.4583$ and $r_1 = -0.1481$. So the final RST controller is:

$$\begin{aligned} R(q^{-1}) &= 0.4583 - 0.1481q^{-1} \\ S(q^{-1}) &= 0.5 - 0.4q^{-1} \\ T(q^{-1}) &= 1 - 0.7417q^{-1} + 0.2019q^{-2} \end{aligned}$$

2. To add a fixed term $H_R(q^{-1}) = 1 + q^{-1}$ in the controller, we should solve the following Bezout equation:

$$A(q^{-1})S'(q^{-1}) + q^{-1}(1 + q^{-1})R(q^{-1}) = P_d(q^{-1})$$

In order to have a solution $n_{S'} = 1$ and so $S'(q^{-1}) = 1 + s'_1q^{-1}$, therefore:

$$(1 - 1.2q^{-1} + 0.35q^{-2})(1 + s'_1q^{-1}) + q^{-1}(1 + q^{-1})(r_0 + r_1q^{-1}) = 1 - 0.7417q^{-1} + 0.2019q^{-2}$$

which leads to the following system of linear equations:

$$\begin{aligned} -1.2 + s'_1 + r_0 &= -0.7417 \\ -1.2s'_1 + 0.35 + r_0 + r_1 &= 0.2019 \quad \Rightarrow \quad s'_1 = 0.2378, r_0 = 0.2205, r_1 = -0.0832 \\ 0.35s'_1 + r_1 &= 0 \end{aligned}$$

So the final RST controller is:

$$\begin{aligned} R(q^{-1}) &= (0.2205 - 0.0832q^{-1})(1 + q^{-1}) = 0.2205 + 0.1373q^{-1} - 0.0832q^{-2} \\ S(q^{-1}) &= (1 + 0.2378q^{-1})(0.5 - 0.4q^{-1}) = 0.5 - 0.2811q^{-1} - 0.0951q^{-2} \\ T(q^{-1}) &= 1 - 0.7417q^{-1} + 0.2019q^{-2} \end{aligned}$$

3. Adding the fixed term can be done using the Q-parameterization as well. Let's take $R_0(q^{-1}) = 0.4583 - 0.1481q^{-1}$ and $S_0(q^{-1}) = 0.5 - 0.4q^{-1}$ and $Q(q^{-1}) = q_0$, then:

$$\begin{aligned} R(q^{-1}) &= R_0(q^{-1}) + q_0A(q^{-1}) = 0.4583 - 0.1481q^{-1} + q_0(1 - 1.2q^{-1} + 0.35q^{-2}) \\ S(q^{-1}) &= S_0(q^{-1}) - q_0B(q^{-1}) = 0.5 - 0.4q^{-1} - q_0(0.5q^{-1} - 0.4q^{-2}) \\ R(-1) &= 0 \quad \Rightarrow \quad 0.4583 + 0.1481 + q_0(1 + 1.2 + 0.35) = 0 \quad \Rightarrow \quad q_0 = -0.2378 \end{aligned}$$

Note that the last equation $R(-1) = 0$ guarantees the existence of $1 + q^{-1}$ in $R(q^{-1})$. Replacing $q_0 = -0.2378$ in R and S gives:

$$\begin{aligned} R(q^{-1}) &= 0.4583 - 0.1481q^{-1} - 0.2378(1 - 1.2q^{-1} + 0.35q^{-2}) = 0.2205 + 0.1373q^{-1} - 0.0832q^{-2} \\ S(q^{-1}) &= 0.5 - 0.4q^{-1} + 0.2378(0.5q^{-1} - 0.4q^{-2}) = 0.5 - 0.2811q^{-1} - 0.0951q^{-2} \\ T(q^{-1}) &= 1 - 0.7417q^{-1} + 0.2019q^{-2} \end{aligned}$$

Problem 3.5: Consider the following plant model:

$$G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} = \frac{0.5q^{-1} - 0.4q^{-2}}{1 - 1.2q^{-1} + 0.35q^{-2}}$$

Take $P_d(q^{-1}) = 1 - 1.2q^{-1} + 0.35q^{-2}$ (choosing closed-loop poles equal to the poles of the plant model is called internal model control) and the same dynamics for tracking and regulation. Design an RST controller that includes an integrator. Note that R should include the polynomial $A(q^{-1})$ in order that the Bezout equation has a solution.

Solution: The Bezout equation is:

$$A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1}) = A(q^{-1})$$

We take $R(q^{-1}) = R'(q^{-1})A(q^{-1})$ and simplify the Bezout equation as follows:

$$S(q^{-1}) + B(q^{-1})R'(q^{-1}) = 1$$

This equation has many solutions, choosing $R'(q^{-1}) = r'_0$ (to obtain the lowest order solutions), we obtain:

$$S(q^{-1}) = 1 - B(q^{-1})r'_0 = 1 - (0.5q^{-1} - 0.4q^{-2})r'_0$$

For any value of r'_0 we have a valid solution. In order to have an integrator in the controller, we pose $S(1) = 0$ that leads to:

$$1 - 0.5r_0 + 0.4r'_0 = 0 \quad \Rightarrow \quad r'_0 = 10$$

Therefore:

$$\begin{aligned} R(q^{-1}) &= R'(q^{-1})A(q^{-1}) = 10 - 12q^{-1} + 3.5q^{-2} \\ S(q^{-1}) &= 1 - 5q^{-1} + 4q^{-2} \\ T(q^{-1}) &= R(1) = 1.5 \end{aligned}$$

Problem 3.6: Consider the following plant model

$$G(z) = \frac{2}{z^2 - 1.6z + 0.7}$$

with $h = 0.2s$.

(a) Design an RST controller such that the regulation dynamic has a natural frequency of $\omega_n = 2$ and a damping factor of $\zeta = 0.7$.

(b) Add the fixed terms $H_R(q^{-1}) = 1 + q^{-1}$ and $H_S(q^{-1}) = 1 - q^{-1}$ in the controller using Q parameterization.

In both items $T(q^{-1})$ should be computed to have the same dynamics for tracking and regulation.

Solution: The plant model is:

$$G(q^{-1}) = \frac{2q^{-2}}{1 - 1.6q^{-1} + 0.7q^{-2}}$$

We have $n_A = 2, n_B = 2$, therefore, $n_R = n_A - 1 = 1$ and $n_S = 1$. Then, the desired closed-loop polynomial is : $P_d(q^{-1}) = 1 + p_1q^{-1} + p_2q^{-2}$ where

$$\begin{aligned} p_1 &= -2e^{-0.7\times 2\times h}\cos 2h\sqrt{1 - 0.7^2} = -1.45 \\ p_2 &= e^{-2\times 0.7\times 2h} = 0.5712 \end{aligned}$$

The Diophantine equation is:

$$(1 - 1.6q^{-1} + 0.7q^{-2})(1 + s_1 q^{-1}) + 2q^{-2}(r_0 + r_1 q^{-1}) = 1 - 1.45q^{-1} + 0.5712q^{-2}$$

Solving the equation gives:

$$\begin{aligned}
-1.6 + s_1 = -1.45 \quad &\Rightarrow \quad s_1 = 0.15 \\
0.7 - 1.6s_1 + 2r_0 = 0.5712 \quad &\Rightarrow \quad r_0 = 0.0556 \\
0.7s_1 + 2r_1 = 0 \quad &\Rightarrow \quad r_1 = -0.0525
\end{aligned}$$

which leads to:

$$\begin{aligned}
R_0(q^{-1}) &= 0.0556 - 0.0525q^{-1} \\
S_0(q^{-1}) &= 1 + 0.15q^{-1} \\
T_0(q^{-1}) &= P(1)/B(1) = 0.0606
\end{aligned}$$

For adding the fixed term we define $Q(q^{-1}) = q_0 + q_1 q^{-1}$ and $R(-1) = S(1) = 0$ which leads to:

$$\begin{array}{l} R(q^{-1}) = R_0(q^{-1}) + A(q^{-1})Q(q^{-1}) \\ S(q^{-1}) = S_0(q^{-1}) - B(q^{-1})Q(q^{-1}) \end{array} \quad \Rightarrow \quad \begin{array}{l} R(-1) = R_0(-1) + A(-1)[q_0 - q_1] = 0 \\ S(1) = S_0(1) - B(1)[q_0 + q_1] \end{array}$$

$$\Rightarrow \quad \begin{array}{l} 0.1081 + 3.3(q_0 - q_1) = 0 \\ 1.15 - 2(q_0 + q_1) = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} q_0 = 0.2711 \\ q_1 = 0.3039 \end{array}$$

$$\Rightarrow \quad \begin{array}{l} R(q^{-1}) = 0.3267 - 0.1824q^{-1} - 0.2964q^{-2} + 0.2127q^{-3} \\ S(q^{-1}) = 1 + 0.15q^{-1} - 0.5422q^{-2} - 0.6078q^{-3} \end{array}$$

Then $T(q^{-1}) = R(1) = 0.0606$.

Problem 3.7: Show that for minimizing the criterion with constant forgetting factor:

$$J(k) = \sum_{i=1}^{k} \lambda_1^{(k-i)} [y(i) - \hat{\theta}^T(k)\phi(i-1)]^2$$

the following PAA can be used.

$$\begin{aligned}
F^{-1}(k+1) &= \lambda_1 F^{-1}(k) + \phi(k)\phi^T(k) \\
\hat{\theta}(k+1) &= \hat{\theta}(k) + F(k+1)\phi(k)\epsilon^{\circ}(k+1)
\end{aligned}$$

Solution: We should compute the gradient of the criterion and put it equal to zero:

$$\frac{\partial J(k)}{\partial \hat{\theta}(k)} = -2\sum_{i=1}^{k} \lambda_1^{(k-i)} [y(i) - \hat{\theta}^T(k)\phi(i-1)]\phi(i-1) = 0$$

which leads to:

$$\left[\sum_{i=1}^{k} \lambda_1^{(k-i)} \phi(i-1)]\phi^T(i-1)\right] \hat{\theta}(k) = \sum_{i=1}^{k} \lambda_1^{(k-i)} y(i)\phi(i-1)$$

Taking

$$F(k) = \left[\sum_{i=1}^{k} \lambda_1^{(k-i)} \phi(i-1)\phi^T(i-1)\right]^{-1}$$

we obtain:

$$\hat{\theta}(k) = \left[\sum_{i=1}^{k} \lambda_1^{(k-i)} \phi(i-1)\phi^T(i-1)\right]^{-1} \sum_{i=1}^{k} \lambda_1^{(k-i)} y(i)\phi(i-1) = F(k) \sum_{i=1}^{k} \lambda_1^{(k-i)} y(i)\phi(i-1)$$

In order to have a recursive algorithm, the computation of $F(k+1)$ is considered:

$$\begin{aligned} F^{-1}(k+1) &= \sum_{i=1}^{k+1} \lambda_1^{(k+1-i)} \phi(i-1)\phi^T(i-1) \\ &= \sum_{i=1}^{k} \lambda_1^{(k+1-i)} \phi(i-1)\phi^T(i-1) + \phi(k)\phi^T(k) \\ &= \lambda_1 \sum_{i=1}^{k} \lambda_1^{(k-i)} \phi(i-1)\phi^T(i-1) + \phi(k)\phi^T(k) = \lambda_1 F^{-1}(k) + \phi(k)\phi^T(k) \end{aligned}$$

A recursive equation for $\hat{\theta}(k+1)$ can also be considered similar to the case without forgetting factor as follows:

$$\begin{aligned} \hat{\theta}(k+1) &= F(k+1) \sum_{i=1}^{t+1} \lambda_1^{(k+1-i)} y(i)\phi(i-1) \\ &= F(k+1) \left\{ \sum_{i=1}^{t} \lambda_1^{(k+1-i)} y(i)\phi(i-1) + y(k+1)\phi(k) \right\} \\ &= F(k+1) \left\{ \lambda_1 F^{-1}(k)\hat{\theta}(k) + y(k+1)\phi(k) \right\} \\ &= F(k+1) \left\{ \left[F^{-1}(k+1) - \phi(k)\phi^T(k)\right] \hat{\theta}(k) + y(k+1)\phi(k) \right\} \\ &= F(k+1) \left\{ F^{-1}(k+1)\hat{\theta}(k) + \phi(k)[y(k+1) - \hat{\theta}^T(k)\phi(k)] \right\} \\ &= \hat{\theta}(k) + F(k+1)\phi(k)\epsilon^\circ(k+1) \end{aligned}$$

Problem 3.8: In a PAA the parameters are updated using:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)\epsilon^\circ(k+1)$$

Show that it can be reformulated as:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\epsilon(k+1)$$

where

$$\epsilon(k+1) = y(k+1) - \hat{\theta}^T(k+1)\phi(k) = \frac{\epsilon^\circ(k+1)}{1 + \phi^T(k)F(k)\phi(k)}$$

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{1 + \phi^T(k)F(k)\phi(k)}$$

Solution: See page 139 of the course notes.

Problem 3.9: Consider the unknown discrete-time model of a plant as:

$$y(k+1) = -ay(k) + bu(k)$$

Compute the parameters of an RST indirect adaptive controller based on pole placement (same dynamics for tracking and regulation) after one sampling period (using $\hat{\theta}(1)$ estimated by recursive LS algorithm with decreasing gain).

The following data are measured : $u(0) = 2, y(0) = 1, y(1) = 5.5$. Assume $\hat{\theta}(0) = [0\,,\,0]^T$, $F(0) = 2I$, $P_d(q^{-1}) = 1 - 0.8q^{-1}$.

Solution: We have

$$\hat{\theta}^T(0) = [0\,,\,0], \qquad \phi^T(0) = [-1\,,\,2], \qquad \hat{y}^\circ(1) = \hat{\theta}^T(0)\phi(0) = 0, \qquad \epsilon^\circ(1) = y(1) - \hat{y}^\circ(1) = 5.5$$

Therefore:

$$F(1) = F(0) - \frac{F(0)\phi(0)\phi^T(0)F(0)}{1 + \phi^T(0)F(0)\phi(0)} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \frac{1}{11}\begin{bmatrix} 4 & -8 \\ -8 & 16 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}$$

The vector of parameter estimates after one sampling period is:

$$\hat{\theta}(1) = \hat{\theta}(0) + F(1)\phi(0)\epsilon^\circ(1) = \frac{1}{11}\begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \times 5.5 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Therefore:

$$\hat{G}(q^{-1}) = \frac{2q^{-1}}{1 - q^{-1}} \quad \Rightarrow \quad n_R = 0 \quad \text{and} \quad n_S = 0$$

Hence $\hat{R}(q^{-1}) = \hat{r}_0$ and $\hat{S}(q^{-1}) = 1$. The Diophantine equation is:

$$(1 - q^{-1}) + 2q^{-1}\hat{r}_0 = 1 - 0.8q^{-1} \quad \Rightarrow \quad (2\hat{r}_0 - 1) = -0.8 \quad \Rightarrow \quad \hat{r}_0 = 0.1$$

Finally $T(q^{-1}) = P(1)/\hat{B}(1) = 0.1$.