Some Exercises for Chapter 2 of Advanced Control Systems

Problem 2.1: Let A be an $m \times n$ matrix, $b \in R^m$. Then show the set of all solutions of $Ax = b$ is a convex subset of R^n.

Solution: Let's take x_1 and x_2 as two solutions of $Ax = b$, i.e. $Ax_1 = b$ and $Ax_2 = b$. Then we should show that $\lambda x_1 + (1-\lambda)x_2$ is also a solution of $Ax = b$. We have:

$$A(\lambda x_1 + (1-\lambda)x_2) = \lambda A x_1 + (1-\lambda)Ax_2 = \lambda b + (1-\lambda)b = b$$

that shows the convexity of the set.

Problem 2.2: Are the following functions convex:

a) $f(x) = e^{ax}$ for $x \in R$ and $a \in R$.

b) $f(x) = x^T Ax + cx$ where $x \in R^n$ and $A = A^T$ is positive.

c) $f(x) = \log(x)$ where $x \in R_+$.

d) $f(x) = \max(x)$ where $x \in R^n$.

e) $f(x) = (x_1 x_2)^{-1}$ where $x \in R^2$ and $x_1 > 0$ and $x_2 > 0$.

f) $f(x) = x_1 x_2 (x_1 - x_2)^{-1}$ where $x \in R^2$ and $x_1 - x_2 > 0$.

g) $f(x) = f_1(x) f_2(x)$ where $f_1(x)$ and $f_2(x)$ are convex.

Solution: A twice differentiable function is convex iff its second derivative is positive.

a) $f(x) = e^{ax}$ is convex because:

$$f'(x) = ae^{ax} \qquad \text{and} \qquad f''(x) = a^2 e^{ax} > 0$$

b) $f(x) = x^T Ax + cx$ is convex because:

$$\nabla f(x) = 2Ax + c \qquad \text{and} \qquad \nabla^2 f(x) = 2A \succ 0$$

c) $f(x) = \log(x)$ is not convex because:

$$f'(x) = \frac{1}{x} \qquad \text{and} \qquad f''(x) = \frac{-1}{x^2} < 0$$

d) $f(x) = \max(x)$ is convex. Lets take two points in R^n, namely x_1 and x_2. The function is convex iff

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

We have $f(\lambda x_1 + (1-\lambda)x_2) = \max(\lambda x_1 + (1-\lambda)x_2)$ and

$$\max(\lambda x_1 + (1-\lambda)x_2) \le \max(\lambda x_1) + \max((1-\lambda)x_2) = \lambda \max(x_1) + (1-\lambda)\max(x_2)$$

So the max function is convex.

e) $f(x) = (x_1x_2)^{-1}$ is convex because:

$$\nabla f(x) = \begin{bmatrix} \frac{-1}{x_1^2x_2} & \frac{-1}{x_1x_2^2} \end{bmatrix}^T \qquad \text{and} \qquad \nabla^2 f(x) = \frac{1}{x_1x_2} \begin{bmatrix} \frac{2}{x_1^2} & \frac{1}{x_1x_2} \\ \frac{1}{x_1x_2} & \frac{2}{x_2^2} \end{bmatrix} \succ 0$$

f) $f(x) = x_1x_2(x_1 - x_2)^{-1}$ is convex because:

$$\nabla f(x) = \begin{bmatrix} \frac{-x_2^2}{(x_1-x_2)^2} & \frac{x_1^2}{(x_1-x_2)^2} \end{bmatrix}^T \qquad \text{and} \qquad \nabla^2 f(x) = \frac{1}{(x_1-x_2)^3} \begin{bmatrix} 2x_2^2 & -2x_1x_2 \\ -2x_1x_2 & 2x_1^2 \end{bmatrix} \succeq 0$$

g) $f(x) = f_1(x)f_2(x)$ is not necessarily convex. As a counterexample, take $f_1(x) = x$ and $f_2(x) = -x$ that are both linear and so convex, then $f(x) = -x^2$ is not convex.

Problem 2.3: Consider an autonomous discrete-time LTI system $x(k+1) = Ax(k)$. Define a Lyapunov function $V(k) = x^T(k)Px(k)$ with $P \succ 0$. Represent the stability condition of the system by an LMI.

Solution: The system is stable if $V(k+1) - V(k) < 0$. We have:

$$\begin{aligned} V(k+1) - V(k) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &= x^T(k)A^TPAx(k) - x^T(k)Px(k) = x^T(k)[A^TPA - P]x(k) \end{aligned}$$

Therefore, $A^TPA - P$ should be negative definite. The stability condition in LMI form:

$$\begin{bmatrix} P & 0 \\ 0 & P - A^TPA \end{bmatrix} \succ 0$$

Problem 2.4: Consider the following LTI discrete-time system:

$$x(k+1) = Ax(k) + Bu(k)$$

and a state feedback law $u(k) = -Kx(k)$. Find the set of stabilizing controllers represented by an LMI.

Solution: The closed-loop state equation is $x(k+1) = (A - BK)x(k)$ which is stable if there exists $P \succ 0$ such that

$$(A - BK)^TP(A - BK) - P \prec 0$$

which is not an LMI. Let's multiply the matrix inequality from left and right by $L = P^{-1}$:

$$L(A - BK)^TL^{-1}(A - BK)L - L \prec 0 \quad \Rightarrow \quad (LA^T - LK^TB)L^{-1}(AL - BKL) - L \prec 0$$

Now, we define a new variable $Y = KL$ which leads to:

$$(LA^T - Y^TB)L^{-1}(AL - BY) - L \prec 0$$

Using Schur Lemma we obtain the following LMI:

$$\begin{bmatrix} L & AL - BY \\ (AL - BY)^T & L \end{bmatrix} \succ 0$$

So the stabilizing controller will be $K = YL^{-1}$.

Problem 2.5: Consider an LTI discrete-time system $G(z)$ with state-space representation (A, B, C, D). Knowing that the impulse response of the system is $g(k) = CA^{k-1}B$ for $k > 0$ and $g(0) = D = 0$. Show that $\|G\|_2^2 = \text{trace}\ (CLC^T)$, where $L = L^T \succ 0$ is the solution to the following Riccati equation:

$$ALA^T - L + BB^T = 0$$

Write a convex optimization problem using LMIs to compute the $\mathcal{H}_2$ norm of a discrete-time system.

Solution: Using Parseval's relation, we have:

$$\begin{aligned}\|G\|_2^2 &= \text{trace}\sum_{k=1}^{\infty} g(k) * g^T(k) = \text{trace}\sum_{k=1}^{\infty}[CA^{k-1}B][CA^{k-1}B]^T \\ &= \text{trace}\sum_{k=1}^{\infty} CA^{k-1}BB^T[A^{k-1}]^T C^T = \text{trace}\ CLC^T\end{aligned}$$

where

$$L = \sum_{k=1}^{\infty} A^{k-1}BB^T[A^{k-1}]^T = BB^T + ABB^TA^T + A^2BB^T[A^T]^2 + \cdots = BB^T + ALA^T$$

which leads to $ALA^T - L + BB^T = 0$. It can be shown that for stable systems larger L makes the left hand side of the above equality more negative, therefore the $\mathcal{H}_2$ norm can be computed by the following convex optimization problem:

$$\min \text{trace}\ CLC^T$$
$$ALA^T - L + BB^T \preceq 0$$

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n=max(size(A));
gamma=sdpvar(1,1);
L=sdpvar(n,n,'symmetric');
lmi1=A*L*A'-L +B*B' <= 0;
lmi2=C*L*C'- gamma <= 0;
cons=[lmi1 lmi2 L>=0];
options=sdpsettings('solver','mosek');
optimize(cons,gamma,options);
norm2=sqrt(value(gamma))
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Problem 2.6: Consider an LTI discrete-time system $G(z)$ with state-space representation $(A, B, C, 0)$. The objective is to design a state feedback controller such that the sum of the two-norm of the closed loop transfer functions from the input disturbance to the output and to the control signal $(-Kx(k))$ is minimized. Represent this objective as a convex optimization problem.

Solution: The state feedback controller is $u(k) = -Kx(k) + w(k)$, where $w(k)$ is the input disturbance. Let's define $z = [y_1(k)\, y_2(k)]^T$, then the closed-loop equations are:

$$\begin{aligned}x(k+1) &= Ax(k) + B(-Kx(k) + w(k)) = (A - BK)x(k) + Bw(k) \\ y_1(k) &= Cx(k) \quad , \quad y_2(k) = -Kx(k)\end{aligned}$$

Therefore, the following optimization problem should be solved:

$$\min \text{trace}\ CLC^T + \text{trace}\ KLK^T$$

$$(A - BK)L(A - BK)^T - L + BB^T \preceq 0$$

The matrix inequality is not linear, so we rewrite it as:

$$(AL - BKL)L^{-1}(AL - BKL)^T - L + BB^T \preceq 0$$

Let's define $Y = KL$ and apply the Schur lemma:

$$\begin{bmatrix} L - BB^T & AL - BY \\ (AL - BY)^T & L \end{bmatrix} \succeq 0$$

which is an LMI. Lets define $CLC^T \prec \Gamma_1$ and $KLK^T \prec \Gamma_2$. The second inequality can be written as: $\Gamma_2 - YL^{-1}Y^T \succ 0$ which can be converted to an LMI using the Schur lemma:

$$\begin{bmatrix} \Gamma_2 & Y \\ Y^T & L \end{bmatrix} \succ 0$$

Then the convex optimization problem is:

$$\min \quad \text{trace } \Gamma_1 + \text{trace } \Gamma_2$$

$$\begin{bmatrix} L - BB^T & AL - BY \\ (AL - BY)^T & L \end{bmatrix} \succeq 0 \quad , \quad \begin{bmatrix} \Gamma_2 & Y \\ Y^T & L \end{bmatrix} \succ 0 \quad , \quad \Gamma_1 - CLC^T \succ 0$$

Problem 2.7: Consider a state feedback control law as $u(t) = r(t) - Kx(t)$ for a strictly proper system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Write a convex optimization problem for computing K that minimizes the infinity norm of the transfer function between the reference signal $r(t)$ and the tracking error $e(t) = r(t) - y(t)$.

Solution: The closed-loop equations are:

$$\dot{x}(t) = (A - BK)x(t) + Br(t)$$
$$e(t) = -Cx(t) + r(t)$$

Then we apply the bounded real lemma (Lemma 2.3, page 63) for the above closed-loop system with $A_{cl} = A - BK, B_{cl} = B, C_{cl} = -C$ and $D_{cl} = I$ to obtain the following inequality:

$$(A - BK)^T P + P(A - BK) + C^T C + (PB - C^T)R^{-1}(PB - C^T)^T \prec 0$$

with $R = (\gamma^2 I - I)$. Multiplying from left and right with $L = P^{-1}$ leads to:

$$L(A - BK)^T + (A - BK)L + LC^T CL + (B - LC^T)R^{-1}((B - LC^T)^T \prec 0$$

Defining a new variable $Y = KL$ gives:

$$LA^T - Y^T B^T + AL - BY + [LC^T \quad (B - LC^T)] \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}^{-1} \begin{bmatrix} CL \\ (B - LC^T)^T \end{bmatrix} \prec 0$$

Using the Schur lemma, we obtain:

$$\begin{bmatrix} LA^T - Y^T B^T + AL - BY & LC^T & (B - LC^T) \\ CL & -I & 0 \\ (B - LC^T)^T & 0 & -R \end{bmatrix} \prec 0$$

which is an LMI in the variables L and Y and γ^2. By minimizing γ^2, the optimal values of L and Y are obtained and the state feedback controller will be $K = YL^{-1}$.

Problem 2.8: Write a convex optimization problem to find a stabilizing controller that minimizes $\|W_2\mathcal{T}\|_\infty$ in a data-driven setting.

Solution: Minimizing $\|W_2\mathcal{T}\|_\infty$ for stable systems can be represented as:

$$\min \quad \gamma$$

$$[W_2GK(I+GK)^{-1}][W_2GK(I+GK)^{-1}]^* \prec \gamma I \qquad \forall\omega\in\Omega$$

Replacing $K = XY^{-1}$, we obtain:

$$[W_2GX(Y+GX)^{-1}][W_2GX(Y+GX)^{-1}]^* \prec \gamma I \qquad \forall\omega\in\Omega$$

Taking $P = Y + GX$, gives:

$$\gamma I - (W_2GX)(P^*P)^{-1}(W_2GX)^* \succ 0 \qquad \forall\omega\in\Omega$$

Applying QMI lemma leads to the following convex optimization problem:

$$\min \quad \gamma$$

$$\begin{bmatrix} \gamma I & W_2GX \\ (W_2GX)^* & P^*P_c + P_c^*P - P_c^*P_c \end{bmatrix} \succ 0 \qquad \forall\omega\in\Omega$$

where $P_c = Y_c + GX_c$ and $K_c = X_cY_c^{-1}$ is an initial stabilizing controller.

Problem 2.9: Consider the model reference control problem in the $\mathcal{H}_2$ framework as:

$$\min_K \|\mathcal{T} - M\|_2$$

where M is the transfer function matrix of a desired closed-loop system and $\mathcal{T} = GK(I+GK)^{-1}$. Write a convex optimization problem in order to compute a stabilizing controller K in a data-driven setting where only the frequency response of the plant model G is available.

Solution: Minimizing $\|\mathcal{T} - M\|_2$ for stable systems can be represented as:

$$\min \quad \int_\Omega \text{trace}\Gamma(\omega)d\omega$$

$$[GK(I+GK)^{-1} - M][GK(I+GK)^{-1} - M]^* \prec \Gamma(\omega) \qquad \forall\omega\in\Omega$$

Replacing $K = XY^{-1}$, we obtain:

$$[GX(Y+GX)^{-1} - M][GX(Y+GX)^{-1} - M]^* \prec \Gamma(\omega) \qquad \forall\omega\in\Omega$$

Taking $P = Y + GX$, gives:

$$\Gamma(\omega) - (GX - MP)(P^*P)^{-1}(GX - MP)^* \succ 0 \qquad \forall\omega\in\Omega$$

Applying QMI lemma and gridding the frequency $\Omega_N = \{\omega_1, \dots, \omega_N\}$, leads to the following convex optimization problem:

$$\min \sum_{k=1}^{N} \text{trace } \Gamma_k$$

$$\begin{bmatrix} \Gamma_k & GX - MP \\ (GX - MP)^* & P^*P_c + P_c^*P - P_c^*P_c \end{bmatrix} \succ 0 \qquad \forall\omega\in\Omega_N$$

where $P_c = Y_c + GX_c$ and $K_c = X_cY_c^{-1}$ is an initial stabilizing controller.

Problem 2.10: Consider a state feedback control law as $u(t) = r(t) - Kx(t)$ for a strictly proper system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

Compute the set of K that makes the transfer function between $r(t)$ and $e(t) = r(t) - y(t)$ positive real (or passive) in terms of Linear Matrix Inequalities. You can use the positive real lemma given below:

Lemma 1 *The system $H(s)$ with state-space representation (A, B, C, D) and $D + D^T \succ 0$ is positive real (i.e. $H(j\omega) + H^*(j\omega) \succ 0$, $\forall\omega$), iff there exists $P = P^T \succ 0$ such that:*

$$A^T P + PA + C^T C + (PB - C^T)(D + D^T)^{-1}(PB - C^T)^T \prec 0$$

Solution: With $u(t) = r(t) - Kx(t)$ the state-space equations of the closed loop system are:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Br(t) - BKx(t) = (A - BK)x(t) + Br(t)\\ e(t) &= -Cx(t) + r(t)\end{aligned}$$

So the state space model of the closed-loop system is $(A - BK, B, -C, I)$. Using PRL we obtain the following matrix inequality:

$$(A - BK)^T P + P(A - BK) + C^T C + (PB + C^T)(2I)^{-1}(PB + C^T)^T \prec 0$$

Multiplying from left and right by $X = P^{-1}$, we obtain:

$$XA^T - XK^T B^T + AX - BKX + XC^T CX + (B + XC^T)(2I)^{-1}(B + XC^T)^T \prec 0$$

Take $KX = Y$:

$$XA^T - Y^T B^T + AX - BY + [XC^T \quad B + XC^T]\begin{bmatrix} I & 0\\ 0 & 2I\end{bmatrix}^{-1}[XC^T \quad B + XC^T]^T \prec 0$$

Applying Schur lemma we obtain:

$$\begin{bmatrix} XA^T - Y^T B^T + AX - BY & XC^T & B + XC^T\\ CX & -I & 0\\ CX + B^T & 0 & -2I\end{bmatrix} \prec 0$$

The above LMI together with $X \succ 0$ guarantee that the final controller $K = YX^{-1}$ makes the closed-loop system positive real..