Solution of exercises of Chapter 1
Advanced Control Systems

Problem 1.1: Consider the following transfer function:

$$G(s) = \frac{\alpha}{\tau s + 1}, \qquad \alpha > 0 \quad , \quad \tau > 0$$

1. Compute two- and ∞-norm of $G(s)$.
2. Compute one-, two- and ∞-norm of $g(t)$, the unit impulse response of $G(s)$.

Solution: For computing the two-norm of $G(s)$ the residue theorem is used:

$$\|G\|_2^2 = \lim_{s \to -1/\tau} (s + 1/\tau) \frac{\alpha}{\tau s + 1} \frac{\alpha}{-\tau s + 1} = \frac{\alpha^2}{2\tau} \quad \Rightarrow \quad \|G\|_2 = \frac{\alpha}{\sqrt{2\tau}}$$

Since $|Gj\omega|$ is a decreasing function with ω, its maximum happens at $\omega = 0$, Thus, $\|G\|_\infty = \alpha$.
The impulse response of G is $g(t) = \frac{\alpha}{\tau} e^{-t/\tau}$ for $t \geq 0$ therefore:

$$\|g\|_1 = \int_0^\infty \frac{\alpha}{\tau} e^{-t/\tau} dt = -\alpha e^{-t/\tau} \Big|_0^\infty = \alpha$$

$$\|g\|_2^2 = \int_0^\infty \frac{\alpha^2}{\tau^2} e^{-2t/\tau} dt = \frac{-\alpha^2 \tau}{2\tau^2} e^{-t/\tau} \Big|_0^\infty = \frac{\alpha^2}{2\tau}$$

$$\|g\|_\infty = \sup_t |g(t)| = \frac{\alpha}{\tau}$$

Problem 1.2: For $G(s)$ stable and strictly proper, show that $\|g\|_1 < \infty$ and find an inequality relating $\|G\|_\infty$ and $\|g\|_1$.

Solution: Let's represent the stable strictly proper transfer function $G(s)$ by its partial-fraction expansion (with no repeated poles) and compute its impulse response:

$$G(s) = \sum_{i=1}^n \frac{c_i}{s - p_i} \quad \Rightarrow \quad g(t) = \sum_{i=1}^n c_i e^{p_i t} \quad t > 0$$

Let's define $\sigma_i := R_e p_i$.Then, since $\sigma_i < 0$, we obtain:

$$\|g\|_1 \leq \sum_{i=1}^n \int_0^\infty |c_i e^{p_i t}| dt = \sum_{i=1}^n \int_0^\infty |c_i| e^{\sigma_i t} dt = \sum_{i=1}^n \frac{|c_i|}{\sigma_i} e^{\sigma_i t} \Big|_0^\infty = \sum_{i=1}^n \frac{-|c_i|}{\sigma_i} < \infty$$

If the system has repeated poles with multiplicity of ℓ_i, we will obtain the terms like $t^{\ell_i - 1} e^{p_i t}$ in $g(t)$ which have bounded integral as well:

$$\int_0^\infty |t^{\ell_i - 1} e^{p_i t}| dt = \int_0^\infty t^{\ell_i - 1} e^{\sigma_i t} dt = \frac{1}{\sigma_i} t^{\ell_i - 1} e^{\sigma_i t} \Big|_0^\infty - \frac{\ell_i - 1}{\sigma_i} \int_0^\infty t^{\ell_i - 2} e^{\sigma_i t} dt = -\frac{\ell_i - 1}{\sigma_i} \int_0^\infty t^{\ell_i - 2} e^{\sigma_i t} dt$$

Continuing the integration we find that:

$$\int_0^\infty |t^{\ell_i - 1} e^{p_i t}| dt = \text{constant} \times \frac{1}{\sigma_i}$$

Finally, the relationship between the norms is $\|G\|_\infty \leq \|g\|_1$, which can be shown based on the definition of $G(j\omega)$:

$$G(j\omega) = \int_0^\infty g(t) e^{-j\omega t} dt$$

Then

$$|G(j\omega)| \leq \int_0^\infty \left| g(t) e^{-j\omega t} \right| dt = \int_0^\infty |g(t)| \left| e^{-j\omega t} \right| dt = \|g\|_1 \quad \Rightarrow \quad \|G\|_\infty \leq \|g\|_1$$

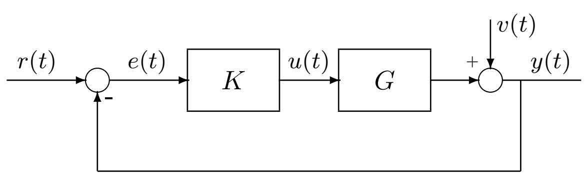
Problem 1.3: Show that the 2-norm for systems is not submultiplicative.

Solution: It is sufficient to provide a counterexample. Take $G(s)$ and $H(s)$ as follows:

$$G(s) = \frac{1}{\tau_1 s + 1} \quad , \quad H(s) = \frac{1}{\tau_2 s + 1}$$

with $\tau_1 = 2\tau_2$. We have $\|G\|_2 = 1/\sqrt{2\tau_1}$ and $\|H\|_2 = 1/\sqrt{2\tau_2} = 1/\sqrt{\tau_1}$. On the other hand it is easy to compute $\|GH\|_2 = 1/\sqrt{3\tau_1}$. A counterexample is obtained for $\tau_1 = 2$. In this case $\|GH\|_2 = 1/\sqrt{6} = 0.4082$, while $\|G\|_2$=0.7071 and $\|H\|_2 = 0.5$.

Problem 1.4: Consider the following feedback loop:



Which system norm should be minimized if the objective is minimizing the 2-norm of the control signal $u(t)$ when $v(t) = 0$ and

1. $r(t)$ is a step signal filtered by a low-pass filter $F(s)$.
2. $r(t)$ is a bounded 2-norm signal whose energy is concentrated between ω_1 and ω_2.

Solution: The transfer function between the reference signal $r(t)$ and the control signal $u(t)$ is:

$$\mathcal{U}(s) = \frac{K}{1 + GK}$$

1. According to Parseval's relation, we should minimize the following system norm:

$$\|u\|_2 = \left\| \frac{1}{s} F(s) \mathcal{U} \right\|_2 = \left\| \frac{FK}{s(1 + GK)} \right\|_2$$

   A necessary condition for the boundedness of the above norm is the existence of a pole at 0 (integrator) in G.

2. Let's define a bandpass filter $W(s)$ such that $|W(j\omega)| = 1$ for $\omega_1 < \omega < \omega_2$ and $|W(j\omega)| = 0$ elsewhere. Then according to Entry (1,1) of Table 1.2 of the course note, we should minimise $\|W\mathcal{U}\|_\infty$.

Problem 1.5: For the unity feedback system with $\tilde{G}(s) = \alpha/s$, where $\alpha \in [-1\,,\,3]$, does there exist a proper controller $K(s)$, such that the system is robustly stable?

Solution: Let's compute a weighting filter that represents the multiplicative uncertainty for the system. Take the nominal model $G(s) = 1/s$, then:

$$\left|\frac{\alpha/j\omega}{1/j\omega} - 1\right| \leq |W_2(j\omega)| \quad \Rightarrow \quad |\alpha - 1| \leq |W_2(j\omega)| \quad \forall\alpha \in [-1\,,\,3] \quad \Rightarrow \quad W_2(s) = 2$$

The robust stability condition is $\|W_2\mathcal{T}\|_\infty < 1$, which leads to $\|\mathcal{T}\|_\infty < 0.5$. However, we have

$$\mathcal{T}(s) = \frac{GK}{1+GK} = \frac{\alpha K}{s+\alpha K} \quad \Rightarrow \quad \mathcal{T}(0) = 1$$

Therefore, the robust stability condition cannot be achieved with any controller.

Problem 1.6: Consider the unity feedback system with

$$\tilde{G}(s) = \frac{1}{s-a} \qquad ; \qquad K(s) = 10$$

where a is real.

1. Convert parametric uncertainty to frequency-domain uncertainty using a feedback uncertainty model. Compute the range of a to have robust stability using the small gain theorem.
2. Compute the range of a for internal stability by analysing the closed-loop poles. Compare it with the result in the previous item.

Solution:

1. Note that the nominal model ($a = 0$), $G(s) = 1/s$, and $K(s) = 10$ is internally stable. We can use the feedback uncertainty model as:

   $$\tilde{G}(s) = \frac{1}{s-a} = \frac{1}{s-a\Delta} = \frac{1/s}{1-a\Delta/s} = \frac{G(s)}{1+\Delta W_2(s)G(s)}$$

   Suppose that $a \geq 0$. Allow Δ to be a variable real number in the interval $[0\,,\,1]$. Then

   - G and $\tilde{G}$ have the same number of poles in Re $s \geq 0$, i.e. $\tilde{G}$ has exactly one pole there.
   - $\|\Delta\|_\infty \leq 1$ and $W_2(s) = -a$.

   The stability condition is $\|W_2G\mathcal{S}\|_\infty < 1$, which gives:

   $$\|W_2G\mathcal{S}\|_\infty = \left\|-a\frac{1/s}{1+10/s}\right\|_\infty = \left\|\frac{-a}{s+10}\right\|_\infty$$

   Since $|-a/(j\omega+10)|$ is a decreasing function, its maximum happens at $\omega = 0$, which leads to $0 \leq a < 10$.

2. The closed loop polynomial is $10 + s - a = 0$ and the closed loop pole is $a - 10$. The system is internally stable for $a < 10$. This range is bigger than those obtained by small gain theorem because of conservatism originated from the choice of Δ. Note that small gain theorem has no conservatism for all possible Δ and has conservatism for any particular choice of Δ.

Problem 1.7: Given a unity feedback control system with

$$G(s) = \frac{s-1}{s^2-4}$$

and the following performance filter:

$$W_1(s) = \frac{\beta}{s+1}$$

find for which values of $\beta > 0$ the nominal performance $\|W_1\mathcal{S}\|_\infty < 1$ can be achieved.

Solution: The model has one pole and one zero in RHP: $z = 1$ and $p = 2$. We now that:

$$\|W_1\mathcal{S}\|_\infty \geq \left|W_1(z)\frac{z+p}{z-p}\right| = \left|\frac{\beta}{2}\frac{1+2}{1-2}\right| \quad \Rightarrow \quad \frac{3\beta}{2} < 1 \quad \Rightarrow \quad \beta < 0.6666$$

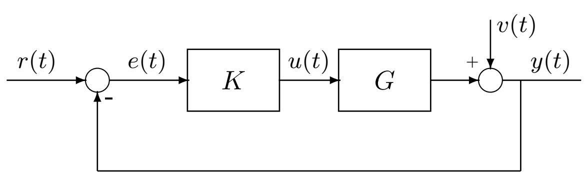
Problem 1.8: The following multiplicative uncertainty model is given:

$$\tilde{G}(s) = G(s)[1 + \Delta W_2(s)] \quad ; \quad G(s) = \frac{s+0.5}{s(s-1)} \quad ; \quad W_2(s) = \frac{s+1}{0.1s+1}$$

Is the unity feedback closed-loop system robustly stabilizable?

Solution: We have two poles with $Res \geq 0$: $p = 0, 1$. Let's compute $W_2(1) = 2/1.1 = 1.818$. Since $\|W_2\mathcal{T}\|_\infty \geq |W_2(p)| = 1.818$, the robust stability condition $\|W_2\mathcal{T}\|_\infty < 1$ cannot be achieved.

Problem 1.9: Consider the following closed-loop system:



where

$$G(s) = \frac{3}{s-\alpha} \qquad \text{and} \qquad K(s) = 4\left(1 + \frac{1}{s}\right)$$

- Assume $\alpha = 5$:
  - **(a)** Compute the two-norm of the tracking error $e(t)$ for a unit step reference signal (assume $v(t) = 0$).
  - **(b)** Compute the infinity norm of the input sensitivity function (the transfer function between the reference signal r and u).
  - **(c)** Assume $v(t) = \sin 2t$. Compute the infinity norm of $u(t)$ when $r(t) = 0$.
- Assume $\alpha \in [3\,,\,7]$:
  - **(d)** Convert parametric uncertainty to frequency-domain uncertainty using a feedback uncertainty model.
  - **(e)** Does the given controller robustly stabilise the closed-loop system?

Solution:

(a) The two norm of $e(t)$ is equal to the two norm of $E(s)$:

$$\|e\|_2^2 = \left\| \frac{1}{1+GK} \frac{1}{s} \right\|_2^2 = \left\| \frac{s(s-5)}{s^2+7s+12} \frac{1}{s} \right\|_2^2 = \left\| \frac{(s-5)}{(s+3)(s+4)} \right\|_2^2 = R_1 + R_2$$

$$R_1 = \lim_{s \to -3} (s+3) \frac{s-5}{(s+3)(s+4)} \frac{-s-5}{(-s+3)(-s+4)} = \frac{8}{21} = 0.381$$

$$R_2 = \lim_{s \to -4} (s+4) \frac{s-5}{(s+3)(s+4)} \frac{-s-5}{(-s+3)(-s+4)} = \frac{-9}{56} = -0.1607$$

$$\|e\|_2^2 = 0.22 \quad \Rightarrow \quad \|e\|_2 = 0.4693$$

(b) The infinity norm of the input sensitivity function is:

$$\|\mathcal{U}(s)\|_\infty = \left\| \frac{K}{1+GK} \right\|_\infty = \left\| \frac{4(s+1)(s-5)}{(s+3)(s+4)} \right\|_\infty = |\mathcal{U}(j\omega)|_{\omega=\infty} = 4$$

Note that by looking at the Bode magnitude diagram, it is clear that the maximum happens at very high frequencies.

(c) $u(t)$ will be a sinusoidal signal with the same frequency and its amplitude (its infinity norm) is:

$$\|u\|_\infty = |U(j2)| = \left| \frac{4(j2+1)(j2-5)}{(j2+3)(j2+4)} \right| \approx 3$$

(d) The uncertain model can be written as (with $-1 \leq \Delta \leq 1$):

$$\tilde{G}(s) = \frac{3}{s-(5-2\Delta)} = \frac{\frac{3}{s-5}}{1+\Delta \frac{2}{3} \frac{3}{s-5}} = \frac{G(s)}{1+\Delta W_2(s) G(s)} \qquad \|\Delta\|_\infty \leq 1 \quad \Rightarrow \quad W_2(s) = 2/3$$

(e) The robust stability condition is $\|W_2 G \mathcal{S}\|_\infty < 1$:

$$\left\| \frac{2}{3} \frac{3}{s-5} \frac{s(s-5)}{(s^2+7s+12)} \right\|_\infty \approx \left| \frac{2j3}{(j3+3)(j3+4)} \right| = \frac{\sqrt{2}}{5} < 1$$

The system has one zero and two poles, so the maximum of the magnitude happens before the second pole ($\omega = 3$). The closed-loop system is robustly stable.