Some Exercises for Chapter 2 of Advanced Control Systems

Problem 2.1: Let A be an $m \times n$ matrix, $b \in R^m$. Then show the set of all solutions of $Ax = b$ is a convex subset of R^n.

Problem 2.2: Are the following functions convex:

a) $f(x) = e^{ax}$ for $x \in R$ and $a \in R$.

b) $f(x) = x^T Ax + cx$ where $x \in R^n$ and $A = A^T$ is positive.

c) $f(x) = \log(x)$ where $x \in R_+$.

d) $f(x) = \max(x)$ where $x \in R^n$.

e) $f(x) = (x_1 x_2)^{-1}$ where $x \in R^2$ and $x_1 > 0$ and $x_2 > 0$.

f) $f(x) = x_1 x_2 (x_1 - x_2)^{-1}$ where $x \in R^2$ and $x_1 - x_2 > 0$.

g) $f(x) = f_1(x) f_2(x)$ where $f_1(x)$ and $f_2(x)$ are convex.

Problem 2.3: Consider an autonomous discrete-time LTI system $x(k+1) = Ax(k)$. Define a Lyapunov function $V(k) = x^T(k)Px(k)$ with $P \succ 0$. Represent the stability condition of the system by an LMI.

Problem 2.4: Consider the following LTI discrete-time system:

$$x(k+1) = Ax(k) + Bu(k)$$

and a state feedback law $u(k) = -Kx(k)$. Find the set of stabilizing controllers represented by an LMI.

Problem 2.5: Consider an LTI discrete-time system $G(z)$ with state-space representation (A, B, C, D). Knowing that the impulse response of the system is $g(k) = CA^{k-1}B$ for $k > 0$ and $g(0) = D = 0$. Show that $\|G\|_2^2 = \text{trace}\ (CLC^T)$, where $L = L^T \succ 0$ is the solution to the following Riccati equation:

$$ALA^T - L + BB^T = 0$$

Write a convex optimization problem using LMIs to compute the two-norm of a discrete-time system.

Problem 2.6: Consider an LTI discrete-time system $G(z)$ with state-space representation $(A, B, C, 0)$. The objective is to design a state feedback controller such that the sum of the two-norm of the closed loop transfer functions from the input disturbance to the output and to the control signal $(-Kx(k))$ is minimized. Represent this objective as a convex optimization problem.

Problem 2.7: Consider a state feedback control law as $u(t) = r(t) - Kx(t)$ for a strictly proper system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Write a convex optimization problem for computing K that minimizes the infinity norm of the transfer function between the reference signal $r(t)$ and the tracking error $e(t) = r(t) - y(t)$.

Problem 2.8: Write a convex optimization problem to find a stabilizing controller that minimizes $\|W_2\mathcal{T}\|_\infty$ in a data-driven setting.

Problem 2.10: Consider the model reference control problem in the $\mathcal{H}_2$ framework as:

$$\min_K \|\mathcal{T} - M\|_2$$

where M is the transfer function matrix of a desired closed-loop system and $\mathcal{T} = GK(I + GK)^{-1}$. Write a convex optimization problem in order to compute a stabilizing controller K in a data-driven setting where only the frequency response of the plant model G is available.

Problem 2.10: Consider a state feedback control law as $u(t) = r(t) - Kx(t)$ for a strictly proper system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Compute the set of K that makes the transfer function between $r(t)$ and $e(t) = r(t) - y(t)$ positive real (or passive) in terms of Linear Matrix Inequalities. You can use the positive real lemma given below:

Lemma 1 *The system $H(s)$ with state-space representation (A, B, C, D) and $D + D^T \succ 0$ is positive real (i.e. $H(j\omega) + H^*(j\omega) \succ 0$, $\forall\omega$), iff there exists $P = P^T \succ 0$ such that:*

$$A^TP + PA + C^TC + (PB - C^T)(D + D^T)^{-1}(PB - C^T)^T \prec 0$$