Robust Controller Design

Outline:

- Introduction to Convex Optimization
  - Convex sets and convex functions
  - Linear Matrix Inequalities (LMIs)
  - Convex optimization
- $\mathcal{H}_2/\mathcal{H}_\infty$ Control (model-based)
  - $\mathcal{H}_2/\mathcal{H}_\infty$ state feedback control
  - Linear Fractional Transformation (LFT)
  - $\mathcal{H}_\infty$ control with a simulation example
- Robust Loopshaping by Convex Optimization (data-driven)
  - Plant and controller structure
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  - Loopshaping and $\mathcal{H}_2/\mathcal{H}_\infty$ control
  - Stability theorem
  - Practical issues
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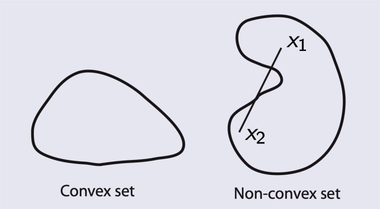
Introduction to Convex Optimization

The main property of a convex optimization problem is that any local minimum is a global minimum.

Definition (Convex set)

A set $\mathbb{S}$ in a vector space is said to be convex if the line segment between any two points of the set lies inside the set.

$$x_1, x_2 \in \mathbb{S} \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in \mathbb{S} \qquad \forall \lambda \in [0 \quad 1]$$



Introduction to Convex Optimization

Definition (Convex Combination)

Let $x_1, \ldots, x_n \in \mathbb{S}$ then :

$$x = \sum_{i=1}^{n} \lambda_i x_i \qquad \text{with} \quad \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1$$

is called a convex combination of $x_1, \ldots, x_n$.

Definition (Convex hull)

For any set $\mathbb{S}$ in a vector space the convex hull consists of all convex combinations of the elements of $\mathbb{S}$ and is a convex set.

Properties of convex sets

Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two convex sets. Then

- $\alpha\mathbb{S}_1 = \{x | x = \alpha c, c \in \mathbb{S}_1\}$ is convex for any scalar α.
- $\mathbb{S}_1 + \mathbb{S}_2 = \{x | x = c_1 + c_2, c_1 \in \mathbb{S}_1, c_2 \in \mathbb{S}_2\}$ is convex.
- $\mathbb{S}_1 \cap \mathbb{S}_2$ is convex but $\mathbb{S}_1 \cup \mathbb{S}_2$ is not necessarily convex.

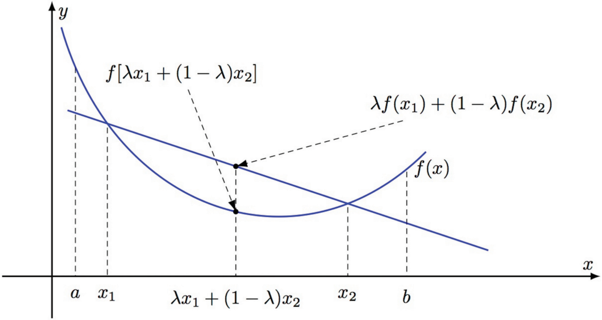
Introduction to Convex Optimization

Definition (Convex function)

A function $f : \mathbb{S} \to \mathbb{R}$ is convex if

1. $\mathbb{S}$ is a convex set and
2. for all $x_1, x_2 \in \mathbb{S}$ and $\lambda \in [0 \quad 1]$ there holds that

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$



Example

Show that $f(x) = x^2$ with $x \in \mathbb{R}$ is a convex function.

Solution: It is clear that $\mathbb{R}$ is a convex set, so we should show for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0 \quad 1]$, it holds that:

$$[\lambda x_1 + (1-\lambda)x_2]^2 \le \lambda x_1^2 + (1-\lambda)x_2^2$$

In order to show the above, we bring all terms to the left side:

$$\lambda^2 x_1^2 + x_2^2 + \lambda^2 x_2^2 - 2\lambda x_2^2 + 2\lambda x_1 x_2 - 2\lambda^2 x_1 x_2 - \lambda x_1^2 - x_2^2 + \lambda x_2^2 \le 0$$

$$(\lambda^2 - \lambda)x_1^2 - 2(\lambda^2 - \lambda)x_1 x_2 + (\lambda^2 - \lambda)x_2^2 \le 0$$

$$(\lambda^2 - \lambda)(x_1 - x_2)^2 \le 0$$

which is always true because $(\lambda^2 - \lambda) \le 0$ for all $\lambda \in [0\,,\, 1]$.

Example

Show that the norm function $f(x) = \|x\|$ with $x \in \mathbb{R}^n$ is a convex function.

Solution: It is clear that $\mathbb{R}^n$ is a convex set, so we should show for all $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in [0 \quad 1]$, it holds that:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

or:

$$\|\lambda x_1 + (1-\lambda)x_2\| \leq \lambda \|x_1\| + (1-\lambda)\|x_2\|$$

In order to show the above, we use the norm properties:

$$\begin{aligned} \|\lambda x_1 + (1-\lambda)x_2\| &\leq \|\lambda x_1\| + \|(1-\lambda)x_2\| \quad \text{(Triangle inequality)} \\ &\leq \lambda \|x_1\| + (1-\lambda)\|x_2\| \quad \text{(Homogeneity)} \end{aligned}$$

Introduction to Convex Optimization

Convex Functions

$f(x) = x^2$ on $\mathbb{R}$, $f(x) = \sin x$ on $[\pi \quad 2\pi]$ and $f(x) = |x|$ on $\mathbb{R}$ are convex, but $f(x) = -x^2$ is not convex.

A twice differentiable function is convex if its domain is convex and its second derivative is non negative.

Example

Is $f(x) = \log(x)$ a convex function $x \in \mathbb{R}^+$?

Solution: No, because $f'(x) = 1/x$ and $f''(x) = -1/x^2 < 0$.

Example

Is $f(x) = (x_1 x_2)^{-1}$ a convex function ($x \in \mathbb{R}^2$ and $x_1 > 0, x_2 > 0$)?

Solution: Yes, because

$$f'(x) = \begin{bmatrix} \frac{-1}{x_1^2 x_2} & \frac{-1}{x_1 x_2^2} \end{bmatrix}^T \qquad \text{and} \qquad f''(x) = \frac{1}{x_1 x_2} \begin{bmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{bmatrix} \succ 0$$

Introduction to Convex Optimization

Convex Functions

1. Linear and affine functions are convex.
2. If $f(y)$ is convex and $y = g(x)$ is linear, $f(g(x))$ is convex.
3. Convex combination of convex functions is also a convex function.

$$g = \sum_{i=1}^{n} \lambda_i f_i \qquad \lambda_i \in [0 \quad 1] \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1$$

is convex if $f_1, \ldots, f_n$ are convex functions.

Example

Is $f(x) = f_1(x) f_2(x)$ a convex function where $f_1(x)$ and $f_2(x)$ are convex functions?

Answer: Not necessarily. As a counterexample, take $f_1(x) = x$ and $f_2(x) = -x$ that are both linear and so convex, but $f(x) = -x^2$ is not convex.

Links between convex sets and convex functions:

1. Epigraph of a convex function (the set of all points lying on or above its graph) is a convex set.
2. If $f(x)$ is a convex function, then
$$\mathbb{D} = \{x \mid f(x) \leq 0\} \text{ is a convex set}$$
3. If $f(x)$ is a linear function on $\mathbb{R}$ then
$$f(x) \leq 0, \quad f(x) \geq 0 \quad \text{and} \quad f(x) = 0 \quad \text{define the convex sets.}$$
4. Let $f_1, \ldots, f_n$ be convex functions then
$$\mathbb{D} = \{x \mid f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, n\}$$
is the intersection of convex sets and defines a convex set.
5. If $f(x)$ is a nonlinear convex function, neither $f(x) \geq 0$ nor $-f(x) \leq 0$ defines a convex set.

Linear Matrix Inequalities

A linear matrix inequality is an expression of the form:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \succ 0$$

- $x = [x_1, \ldots, x_m]$ is a vector of m decision variables,
- $F_i = F_i^T \in \mathbb{R}^{n \times n}, i = 0, \ldots, m$,
- The special inequality $\succ 0$, means positive definite.

Positive definite matrices: Matrix $F \succ 0$ if $u^T F u > 0$ for all $u \in \mathbb{R}^n$ and $u \neq 0$. The following statements are necessary and sufficient for a real symmetric matrix F to be positive definite:

- All eigenvalues of F are positive.
- All principal minors of F (det. of principal submatrices) are positive.

If $F \succ 0$, then its determinant is positive and all values on the main diagonal of F are also positive.

Further properties of real positive definite matrices:

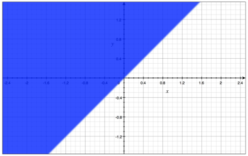
1. Every positive definite matrix is invertible and the inverse is also positive definite.
2. If $F \succ 0$ and $\lambda > 0$ is a real number, then $\lambda F \succ 0$.
3. If $F \succ 0$ and $G \succ 0$ then $F + G \succ 0$ and $GFG \succ 0$ and $FGF \succ 0$ and $\text{tr}(FG) > 0$. The product FG is also positive definite if $FG = GF$.
4. If $F \succ 0$ and M has full rank, then $M^T FM \succ 0$. In the same way. if $F \prec 0$, then $M^T FM \prec 0$.
5. If $F \succ 0$ then there is $\delta > 0$ such that $F \succeq \delta I$ (it means that $F - \delta I \succeq 0$).

Main property: $F(x) \succ 0$ and $F(x) \prec 0$ define convex sets on x.
For example the set $\mathbb{S} = \{x | F(x) \succ 0\}$ is convex. It means if $x_1, x_2 \in \mathbb{S}$ and $\lambda \in [0 \quad 1]$, then:

$$F(\lambda x_1 + (1 - \lambda) x_2) = \lambda F(x_1) + (1 - \lambda) F(x_2) \succ 0$$

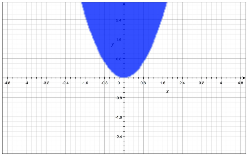
Linear Matrix Inequalities

Many convex sets can be represented by LMI:



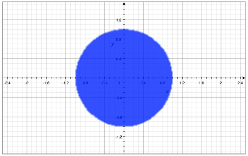
$x_2 > x_1$

$F(x) = [x_2 - x_1] > 0$



$x_2 > x_1^2$

$$F(x) = \begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix} \succ 0$$



$x_1^2 + x_2^2 < 1$

$$F(x) = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix} \succ 0$$

$F(x) \succ 0$ can always be represented as an LMI if its elements are affine w.r.t x.

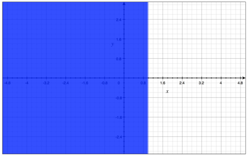
$$F(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Moreover, any matrix inequality which is affine w.r.t $F(x)$ and symmetric is also an LMI (i.e. $AF(x)A^T + BB^T \succ 0$)

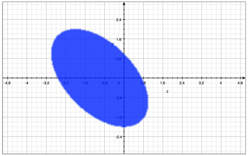
Geometry of LMIs: An LMI is the intersection of constraints on some polynomial functions (the principal minors).

$$F(x) = \begin{bmatrix} 1-x_1 & x_1+x_2 & x_1 \\ x_1+x_2 & 2-x_1 & 0 \\ x_1 & 0 & 1+x_2 \end{bmatrix} \succ 0$$

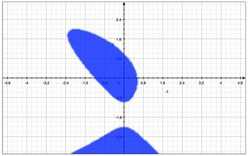
$$\begin{aligned} m_1 &: \quad 1-x_1 > 0 \\ m_2 &: \quad (1-x_1)(2-x_1)-(x_1+x_2)^2 > 0 \\ m_3 &: \quad x_1^2(x_2-2)+(1+x_2)[(1-x_1)(2-x_2)-(x_1+x_2)^2 > 0 \end{aligned}$$



$m_1 > 0$



$m_2 > 0$



$m_3 > 0$

Linear Matrix Inequalities

Lemma

Schur lemma: *If $A = A^T$ and $C = C^T$ then :*

$$F = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \quad \Longleftrightarrow \quad A \succ 0 \quad \text{and} \quad C - B^T A^{-1} B \succ 0$$
$$\Longleftrightarrow \quad C \succ 0 \quad \text{and} \quad A - BC^{-1} B^T \succ 0$$

$$F = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \prec 0 \quad \Longleftrightarrow \quad A \prec 0 \quad \text{and} \quad C - B^T A^{-1} B \prec 0$$
$$\Longleftrightarrow \quad C \prec 0 \quad \text{and} \quad A - BC^{-1} B^T \prec 0$$

$C - B^T A^{-1} B$ and $A - BC^{-1} B^T$ are called Schur complements.

Proof:

$$\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix}$$

Note that $M^T F M$ has the same sign of F if M is nonsingular.

Introduction to Convex Optimization

Convex optimization problem

$$\min_x f_0(x)$$

subject to

$$
\begin{aligned}
f_i(x) &\le 0 \qquad i = 1, \ldots, n \qquad f_i \text{ Convex} \\
g_j(x) &= 0 \qquad j = 1, \ldots, m \qquad g_j \text{ Linear}
\end{aligned}
$$

Linear programming: $f_0(x)$ and $f_i(x)$ are linear.

Quadratic programming: $f_0(x)$ is quadratic, but $f_i(x)$ are linear.

Semidefinite programming: $f_0(x)$ is linear and the constraints are symmetric semidefinite matrices (Linear Matrix Inequalities).

Semi-infinite programming: The constraints are defined for a parameter $\theta \in \Theta$ (number of constraints goes to infinity). This type of problems is called robust optimization.

LMIs in Control

Example (Stability analysis)

A continuous-time LTI autonomous system $\dot{x}(t) = Ax(t)$ is asymptotically stable ($\lim_{t \to \infty} x(t) = 0, \quad \forall x_0 \neq 0$) iff there exists a quadratic Lyapunov function $V(x) = x^T P x$ such that:

$$V(x) > 0 \quad \text{and} \quad \dot{V}(x) < 0$$

These two conditions are verified iff there exists a symmetric matrix $P \succ 0$ such that

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x < 0$$

This is equivalent to the feasibility of the following LMI:

$$\begin{bmatrix} P & 0 \\ 0 & -(A^T P + PA) \end{bmatrix} \succ 0$$

LMIs in Control

Question (Stability of discrete-time systems)

Consider an autonomous discrete-time LTI system $x(k+1) = Ax(k)$. Define a Lyapunov function $V(k) = x^T(k)Px(k)$ with $P \succ 0$. Represent the stability condition by an LMI.

Solution: The system is stable if $V(k+1) - V(k) < 0$. We have:

$$\begin{aligned} V(k+1) - V(k) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &= x^T(k)A^TPAx(k) - x^T(k)Px(k) \\ &= x^T(k)[A^TPA - P]x(k) \end{aligned}$$

Therefore, $A^TPA - P$ should be negative definite. The stability condition in LMI form is:

$$\begin{bmatrix} P & 0 \\ 0 & P - A^TPA \end{bmatrix} \succ 0$$

LMIs in Control

Example (Stability of polytopic systems)

Consider the LTI system $\dot{x}(t) = Ax(t)$ where $A \in \text{co}\{A_1, \dots, A_N\}$.
This system is quadratically stable iff there exists $P \succ 0$ such that :

$$A_i^T P + P A_i \prec 0 \qquad \forall i \in [1 \quad N]$$

Proof: We know that :

$$\text{co}\{A_1, \dots, A_N\} = \left\{ A \,\middle|\, A(\lambda) = \sum_{i=1}^{N} \lambda_i A_i, \quad \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i = 1 \right\}$$

On the other hand : $\sum_{i=1}^{N} \lambda_i (A_i^T P + P A_i) = A^T(\lambda) P + P A(\lambda) \prec 0$

- Stability of a polytopic system is ensured by stability of its vertices.
- Quadratic stability guarantees stability for fast parameter variations.
- Quadratic stability condition is too conservative for robust stability.

LMIs in Control

Stability of polytopic systems

Consider the discrete-time LTI system with polytopic uncertainty as $x(k+1) = A(\lambda)x(k)$ where $A(\lambda) = \sum_{i=1}^{N} \lambda_i A_i$, $\lambda \geq 0$ and $\sum_{i=1}^{N} \lambda_i = 1$. Show that this system is quadratically stable iff there exists $P \succ 0$ such that : $A_i^T P A_i - P \prec 0 \quad \forall i \in [1 \quad N]$.

Proof: The stability condition for A_i can be reformulated using the Schur Lemma as the following matrix inequality:

$$\begin{bmatrix} P & A_i^T \\ A_i & P^{-1} \end{bmatrix} \succ 0$$

If we multiply the inequality by λ_i and take the sum over i, we have:

$$\begin{bmatrix} \sum_{i=1}^{N} \lambda_i P & \sum_{i=1}^{N} \lambda_i A_i^T \\ \sum_{i=1}^{N} \lambda_i A_i & \sum_{i=1}^{N} \lambda_i P^{-1} \end{bmatrix} = \begin{bmatrix} P & A^T(\lambda) \\ A(\lambda) & P^{-1} \end{bmatrix} \succ 0$$

That is equivalent to $A^T(\lambda) P A(\lambda) - P \prec 0$.

LMIs in Control

Bounded Real Lemma

Let $\gamma > 0$ and $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, then $\|G\|_\infty < \gamma$ if and only if there exists $P \succ 0$ such that

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \prec 0$$

Proof:

If u is the input and y the output of G then the infinity norm can be defined as the supremum of the two-norm gain:

$$\|G\|_\infty = \sup_u \frac{\|y\|_2}{\|u\|_2}$$

The state space representation of G is given by:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$

Proof of Bounded Real Lemma

We prove only the sufficient condition. The LMI implies:

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \left\{ \begin{bmatrix} A^T P + PA & PB \\ B^T P & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \right\} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} < 0$$

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$
$$+[\underbrace{Cx(t) + Du(t)}_{y(t)}]^T [Cx(t) + Du(t)] < 0$$

$$\Rightarrow \qquad x^T(A^T P + PA)x + x^T PBu + u^T B^T Px - \gamma^2 u^T u + y^T y < 0$$
$$\Rightarrow \qquad (Ax + Bu)^T Px + x^T P(Ax + Bu) - \gamma^2 u^T u + y^T y < 0$$

LMIs in Control

Proof of Bounded Real Lemma

Taking a Lyapunov function $V(x) = x^T P x$, we have $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$ and:

$$\dot{V}(x) - \gamma^2 u^T u + y^T y < 0$$

Taking the integral of the above inequality we obtain:

$$\int_0^\infty \dot{V}(x)dt - \gamma^2 \int_0^\infty u^T u dt + \int_0^\infty y^T y dt < 0$$

$$V(x(\infty)) - V(x(0)) - \gamma^2 \|u\|_2^2 + \|y\|_2^2 < 0$$

Since $A^T P + PA \prec 0$ from the LMI in the lemma, G is stable and $x(\infty) = x(0) = 0$ that leads to $V(x(\infty)) = V(x(0)) = 0$. Therefore:

$$\frac{\|y\|_2^2}{\|u\|_2^2} < \gamma^2 \quad \forall u \quad \Rightarrow \quad \sup_u \frac{\|y\|_2}{\|u\|_2} < \gamma \quad \Rightarrow \quad \|G\|_\infty < \gamma$$

LMIs in Control

Example (Computing $\mathcal{H}_\infty$ norm by convex optimization)

Using the bounded real lemma for a strictly proper system ($D = 0$), we should minimize γ^2 (the square of the ∞-norm) such that:

$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} \prec 0$$

The nonlinearity in γ^2 can be fixed by change of variable to convert the above inequality to an LMI.

This problem can be solved by the following YALMIP code:

```
gamma2=sdpvar(1,1);
P=sdpvar(n,n,'symmetric');
lmi=[A'*P+P*A+C'*C P*B; B'*P -gamma2*eye(n)]<0;
lmi=[lmi, P>0];
options=sdpsettings('solver','mosek');
optimize(lmi,gamma2,options);
gamma=sqrt(value(gamma2))
```

$\mathcal{H}_\infty$ state feedback control

Problem: Given a controllable representation of the plant model:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

Compute a state feedback controller K such that the $\mathcal{H}_\infty$-norm of the closed-loop system from an input disturbance $v(t)$ to the output is minimized.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ u(t) &= v(t) - Kx(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}(t) &= (A - BK)x(t) + Bv(t) \\ y(t) &= Cx(t) \end{aligned}$$

Solution: Using the bounded real lemma (and applying the Schur lemma) we have :

$$(A - BK)^T P + P(A - BK) + C^T C + PB\gamma^{-2}B^T P \prec 0$$

which is not an LMI.

$\mathcal{H}_\infty$ state feedback control

- Now, multiply the inequality from left and right by $X = \gamma^2 P^{-1}$:

$$X(A - BK)^T \gamma^2 + \gamma^2 (A - BK) X + \gamma^2 BB^T + XC^T CX \prec 0$$

- Denoting $Y = KX$ and multiplying by γ^{-2}, we derive:

$$XA^T + AX - Y^T B^T - BY + BB^T + XC^T \gamma^{-2} CX \prec 0$$

- This matrix inequality can be converted to an LMI using the Schur lemma:

$$\begin{bmatrix} XA^T + AX - Y^T B^T - BY + BB^T & XC^T \\ CX & -\gamma^2 I \end{bmatrix} \prec 0, \quad X \succ 0$$

- After minimizing γ^2 subject to the above LMI constraints the state feedback controller is computed by $K = YX^{-1}$.

Computing $\mathcal{H}_2$ norm

Lemma ($\mathcal{H}_2$ norm by SDP)

Let $G(s)$ be a strictly proper stable transfer function with a state-space representation $(A, B, C, 0)$. Then $\|G\|_2^2 = trace[CL^\circ C^T]$ where L° is the optimal solution to the following SDP problem:

$$\min_{L} \; trace[CLC^T]$$

$$AL + LA^T + BB^T \preceq 0 \qquad ; \qquad L \succ 0 \qquad (1)$$

Proof: Let $L^* \succ 0$ be the unique solution to $AL + LA^T + BB^T = 0$. Since L^* satisfies the equality, it also satisfies the inequality in (1). Therefore, trace$(CL^\circ C^T) \leq$ trace$(CL^* C^T)$, which implies trace$(C(L^\circ - L^*)C^T) \leq 0$ and hence $L^\circ - L^* \preceq 0$. On the other hand, subtracting $AL + LA^T + BB^T = 0$ from (1) yeilds $A(L^\circ - L^*) + (L^\circ - L^*)A^T \preceq 0$. By stability of A, this implies $L^\circ - L^* \succeq 0$. Combining both inequalities, we conclude $L^\circ - L^* = 0$, i.e. $L^\circ = L^*$, which complete the proof.

Computing $\mathcal{H}_2$ norm

Example (Computing $\mathcal{H}_2$ norm by convex optimization)

the 2-norm of a transfer function can be obtained by the following SDP problem:

$$\min_{L} \text{trace}[CLC^T]$$

$$AL + LA^T + BB^T \preceq 0 \qquad ; \qquad L \succ 0$$

This can be coded using Yalmip as follows:

```
L=sdpvar(n,n,'symmetric');
lmi=A*L+L*A'+B*B' <=0;
lmi=[lmi, L>0];
options=sdpsettings('solver','mosek');
optimize(lmi,trace(C*L*C'),options);
L=value(L);
H2Norm=sqrt(trace(C*L*C'))
```

$\mathcal{H}_2$ state feedback control

Problem: Given a controllable representation of the plant model:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

Compute a state feedback controller K such that the two norm of the closed-loop system from the input disturbance to the output is minimized.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ u(t) &= v(t) - Kx(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}(t) &= (A - BK)x(t) + Bv(t) \\ y(t) &= Cx(t) \end{aligned}$$

Solution: The two-norm minimization can be converted to a convex optimization problem.

$$\min tr(CLC^T)$$

$$(A - BK)L + L(A - BK)^T + BB^T \preceq 0 \qquad ; \qquad L \succ 0$$

The inequality can be converted to an LMI denoting $Y = KL$:

$$AL + LA^T - BY - Y^TB^T + BB^T \preceq 0$$

Example (State disturbance rejection:)

Consider a controllable state-space model of a system as:

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 w(t)$$
$$y(t) = Cx(t)$$

Design a state feedback controller that minimizes the infinity-norm of the transfer function between $w(t)$ and $[y(t)\,;\,u(t)]$.

Solution:

- Replace $u(t) = -Kx(t)$ to find the closed-loop state-space model as:

$$\dot{x}(t) = (A - B_1 K)x(t) + B_2 w(t)$$
$$y(t) = Cx(t)$$
$$u(t) = -Kx(t)$$

- Define a new output variable $z(t) = [y(t)\,;\,u(t)]$ as one performance output. The output equation of the closed-loop system will be:

$$z(t) = \begin{bmatrix} C \\ -K \end{bmatrix} x(t)$$

$\mathcal{H}_\infty$ state feedback control

Example (State disturbance rejection:)

- Apply bounded real lemma to the closed-loop state-space model ($P \succ 0$):

$$(A - B_1 K)^T P + P(A - B_1 K) + C^T C + K^T K + P B_2(\gamma^{-2}) B_2^T P \prec 0$$

  Which is not an LMI.

- Multiply from left and right by $X = \gamma^2 P^{-1}$, and define a new variable $Y = KX$ and multiply the whole inequality by γ^{-2} to obtain:

$$XA^T - Y^T B_1^T + AX - B_1 Y + XC^T \gamma^{-2} CX + Y^T \gamma^{-2} Y + B_2 B_2^T \prec 0$$

- Which can be rewritten as:

$$XA^T + AX - Y^T B_1^T - B_1 Y + B_2 B_2^T$$
$$- \begin{bmatrix} XC^T & Y^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} XC^T & Y^T \end{bmatrix}^T \prec 0$$

$\mathcal{H}_\infty$ state feedback control

Example (State disturbance rejection:)

- Which can be rewritten as:

$$XA^T + AX - Y^T B_1^T - B_1 Y + B_2 B_2^T - \begin{bmatrix} XC^T & Y^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} XC^T & Y^T \end{bmatrix}^T \prec 0$$

- Then apply Schur Lemma to find the following LMI:

$$\begin{bmatrix} XA^T + AX - Y^T B_1^T - B_1 Y + B_2 B_2^T & XC^T & Y^T \\ CX & -\gamma^2 I & 0 \\ Y & 0 & -\gamma^2 I \end{bmatrix} \prec 0 \quad , \quad X \succ 0$$

- The final state feedback controller that achieve this performance is $K = YX^{-1}$.

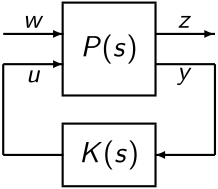
$\mathcal{H}_2/\mathcal{H}_\infty$ output feedback control

- In many systems the states are not all available/measurable.
- The feedback controller has access only to the measurable states (outputs).
- A state observer/estimator should be used.
- Only one transfer function can be minimized.
- The feedback system should be rearranged to consider all robust stability and performance in **ONE** transfer function.
- Linear Fractional Transformation (LFT) is used to rearrange the feedback system.

Linear Fractional Transformation

A feedback control system can be rearranged as an LFT:

- w: all external inputs
- u: control inputs
- z: error signals
- y: measured outputs



$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad u = Ky \quad \Rightarrow \quad \begin{aligned} z &= P_{11}w + P_{12}Ky \\ y &= P_{21}w + P_{22}Ky \end{aligned}$$

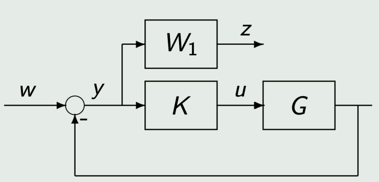
$$T_{zw} = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

- $P(s)$, called augmented plant, includes the plant model $G(s)$ and all weighting filters.
- $T_{zw} = F_l(P, K)$ is the transfer function between the error signals z and external inputs w. In $\mathcal{H}_2/\mathcal{H}_\infty$ control problems the objective is to minimize $\|F_l(P, K)\|$.

Linear Fractional Transformation

Example (Nominal Performance)

Show the nominal performance problem as an LFT (Find the augmented plant P and $F_l(P,K)$):



$$z = W_1(w - Gu)$$
$$y = w - Gu$$

$$\begin{pmatrix} z \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} W_1 & -W_1 G \\ 1 & -G \end{pmatrix}}^{P} \begin{pmatrix} w \\ u \end{pmatrix}$$

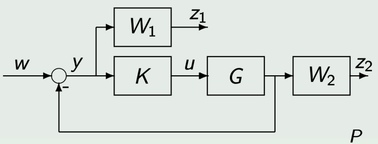
$$\begin{aligned} F_l(P,K) &= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &= W_1 + (-W_1 G)K(1 + GK)^{-1}1 \\ &= W_1(1 - \frac{GK}{1+GK}) = \frac{W_1}{1+GK} = W_1 \mathcal{S} \end{aligned}$$

Remark: The Matlab command `P=augw(G,W1)` will generate the augmented plant.

Linear Fractional Transformation

Example (Robust Performance)

Show the robust performance problem for multiplicative uncertainty as an LFT:



$$
\begin{aligned}
z_1 &= W_1(w - Gu)\\
z_2 &= W_2 G u\\
y &= w - Gu
\end{aligned}
\qquad
\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} W_1 & -W_1 G \\ 0 & W_2 G \\ 1 & -G \end{pmatrix}}^{P} \begin{pmatrix} w \\ u \end{pmatrix}
$$

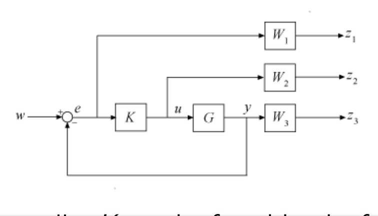
$$
F_l(P,K) = \begin{pmatrix} W_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -W_1 G \\ W_2 G \end{pmatrix} K(1 + GK)^{-1} = \begin{pmatrix} W_1 S \\ W_2 T \end{pmatrix}
$$

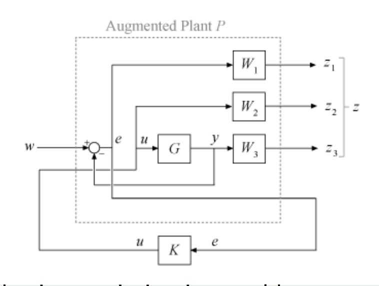
$$
\begin{aligned}
\|F_l(P,K)\|_\infty = \left\| \begin{pmatrix} W_1 \mathcal{S} \\ W_2 \mathcal{T} \end{pmatrix} \right\|_\infty &= \sup_\omega \bar{\sigma} \begin{pmatrix} W_1(j\omega)\mathcal{S}(j\omega) \\ W_2(j\omega)\mathcal{T}(j\omega) \end{pmatrix}\\
&= \sup_\omega \sqrt{|W_1(j\omega)\mathcal{S}(j\omega)|^2 + |W_2(j\omega)\mathcal{T}(j\omega)|^2}
\end{aligned}
$$

Linear Fractional Transformation

Example (Mixed Sensitivity Problem)

In many practical problems we are interested in shaping three closed-loop sensitivity functions





The controller K can be found by the following optimization problem:

$$\min_{K,\gamma} \quad \gamma$$

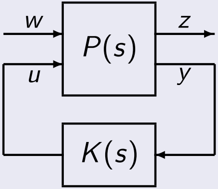
$$\left\| \begin{bmatrix} W_1\mathcal{S} \\ W_2K\mathcal{S} \\ W_3\mathcal{T} \end{bmatrix} \right\|_\infty < \gamma \qquad \Rightarrow \qquad \begin{array}{ll} |\mathcal{S}(j\omega)| < \gamma |W_1^{-1}(j\omega)| & \forall\omega \\ |K(j\omega)\mathcal{S}(j\omega)| < \gamma |W_2^{-1}(j\omega)| & \forall\omega \\ |\mathcal{T}(j\omega)| < \gamma |W_3^{-1}(j\omega)| & \forall\omega \end{array}$$

$\mathcal{H}_\infty$ Control

State-Space LFT Representation

Consider the system described by:

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$



$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned}$$

$\mathcal{H}_\infty$ Control:

Optimal $\mathcal{H}_\infty$ Control: Find all admissible controllers $K(s)$ such that $\|T_{zw}\|_\infty$ is minimized.

Suboptimal $\mathcal{H}_\infty$ Control: Given $\gamma > 0$, find all admissible controllers $K(s)$, if there are any, such that $\|T_{zw}\|_\infty < \gamma$.

$\mathcal{H}_\infty$ Control

Assumptions:

(A1) (A, B_1, C_1) is controllable and observable ; (A, B_2, C_2) is stabilizable and detectable:

(A2) D_{12} has full column rank and D_{21} has full row rank.

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

Assumption (A1)

It is a standard assumption for the existence of a stabilizing controller. Stabilizable and detectable are weaker conditions than controllable and observable. If the uncontrollable states are stable the system is stabilizable and if the unobservable states are stable the system is detectable.

$\mathcal{H}_\infty$ Control (Assumptions)

Assumption (A2)

If D_{12} has not full column rank, it means that some control inputs have no direct effect on the controlled outputs $z = C_1 x + D_{12} u$. This makes the problem singular and cannot be solved. The solution is to add some weighting filters on these control inputs (even a very small gain to avoid the singularity in the computations).

If D_{21} has not full row rank, it means that one of the measured outputs $y = C_2 x + D_{21} w$ is not directly affected by any of external inputs. This makes the problem singular and cannot be solved. The solution is to add some external inputs (noise or disturbance) on all measured output (with a very small gain).

Assumption (A3, A4)

These Assumptions are not satisfied if there are some poles of the plant model or weighting filters on the imaginary axis. This problem can be solved by a small perturbation of the poles on the imaginary axis.

$\mathcal{H}_\infty$ Control

Integrator in the controller

In order to have an integral action in the controller the filter W_1 should include an integrator. In this case, the sensitivity function $\mathcal{S}$ should have a zero at origin to have $\|W_1\mathcal{S}\|_\infty$ bounded then K must have a pole at origin. However, this will violate Assumptions A3 and A4. The remedy is to consider a quasi-integrator in W_1 (a pole very close to zero). This will lead to a quasi integrator in the controller that can be replaced with an integrator.

$D_{22} \neq \mathbf{0}$

This problem occurs when we transform a discrete system to a continuous system. In this case we can solve the problem for $D_{22} = 0$ and compute the controller K_0 and then the final controller is: $K = K_0(I + D_{22}K_0)^{-1}$.

Simplifying Assumption

$$\text{(A5)} \quad D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$\mathcal{H}_\infty$ Control

Theorem

Under Assumptions A1 to A5 for an augmented plant in an LFT representation, there exists a suboptimal controller such that $\|T_{zw}\|_\infty < \gamma$ *as:*

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A} & (I - \gamma^{-2} Y X)^{-1} Y C_2^T \\ \hline -B_2^T X & 0 \end{array}\right]$$

where: $\hat{A} = A + \gamma^{-2} B_1 B_1^T X - B_2 B_2^T X - (I - \gamma^{-2} YX)^{-1} Y C_2^T C_2.$

Moreover, $Y \succ 0$ *and* $X \succ 0$ *are the solutions to the following Riccati equations:*

$$XA + A^T X + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 = 0$$

$$AY + YA^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T = 0$$

that satisfy $\rho(XY) < \gamma^2$ $\quad\rho(\cdot) = |\lambda_{\max}(\cdot)|$*: spectral radius*

$\mathcal{H}_\infty$ Control (example from Matlab toolbox)

Example (Robust control of a system with multimodel uncertainty)

Plant Model: Consider an unstable system $G_n(s) = \dfrac{2}{s-2}$ as the nominal model with the following multimodel uncertainty:

Extra lag:	$G_1(s) = \frac{2}{(0.06s+1)(s-2)}$
Time delay:	$G_2(s) = \frac{2e^{-0.02s}}{(s-2)}$
High frequency resonance:	$G_3(s) = \frac{2}{s-2}\frac{2500}{(s^2+10s+2500)}$
High frequency resonance:	$G_4(s) = \frac{2}{s-2}\frac{4900}{(s^2+28s+4900)}$
Pole/gain migration:	$G_5(s) = \frac{2.4}{(s-2.2)}$
Pole/gain migration:	$G_6(s) = \frac{1.6}{(s-1.8)}$

Control Performance: We should design a controller that stabilizes all plant models and achieves a closed-loop bandwidth of 10 rad/s.

$\mathcal{H}_\infty$ Control (example from Matlab toolbox)

Example (Robust control of a system with multimodel uncertainty)

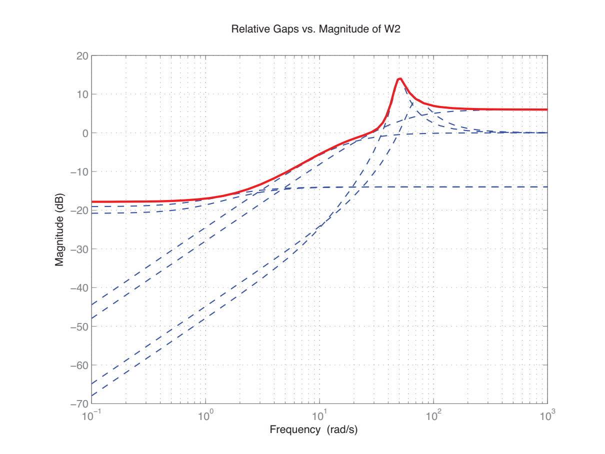
Model Uncertainty: The multimodel uncertainty can be converted to multiplicative uncertainty.

```
s=tf('s');
Gn = 2/(s-2);                          % nominal model
G1 = Gn/(0.06*s+1);                    % extra lag
G2 = Gn*(50-s)/(s+50);                 % time delay (Pade approximation)
G3 = Gn*50^2/(s^2+2*.1*50*s+50^2);     % high frequency resonance
G4 = Gn*70^2/(s^2+2*.2*70*s+70^2);     % high frequency resonance
G5 = 2.4/(s-2.2);                      % pole/gain migration
G6 = 1.6/(s-1.8);                      % pole/gain migration

% command to gather the plant models G1 through G6 into one array.

G = stack(1,G1,G2,G3,G4,G5,G6);

% Try a 4th-order filter W2 for Multiplicative uncertainty:

orderW2 = 4;
Gf = frd(G,logspace(-1,3,60));
[Gu,Info] = ucover(Gf,Gn,orderW2,'InputMult');
W2 = Info.W1;
bodemag((Gn-Gf)/Gn,'b--',W2,'r'); grid
```

$\mathcal{H}_\infty$ Control (example from Matlab toolbox)

Example (Robust control of a system with multimodel uncertainty)

Model Uncertainty: The multimodel uncertainty can be converted to multiplicative uncertainty ($W_2(s)$ is computed).



$\mathcal{H}_\infty$ Control (example from Matlab toolbox)

Example (Robust control of a system with multimodel uncertainty)

***$W_1(s)$* Filter Design:** In order to have a bandwidth of at least ω_b=10 rad/s and a modulus margin of bigger than $m = 0.5$, we choose:

$$W_1^{-1}(s) = \frac{s}{m(s+\omega_b)} = \frac{2s}{s+10} \quad \Rightarrow \quad W_1(s) = \frac{s+10}{2(s+0.00001)}$$

Mixed Sensitivity Design: In order to obtain the robust performance we minimize

$$\|[W_1\mathcal{S} \quad W_2\mathcal{T}]\|_\infty$$

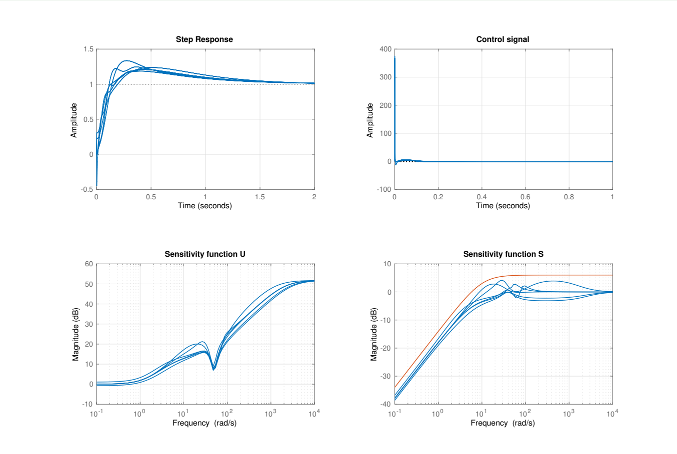
Matlab Code:

```
W1=(s+10)/2/(s+0.000001);
K=mixsyn(Gn,W1,[],W2);
T=feedback(G*K,1);
U=feedback(K,G);
```

$\mathcal{H}_\infty$ Control (example from Matlab toolbox)

Example (Robust control of a system with multimodel uncertainty)

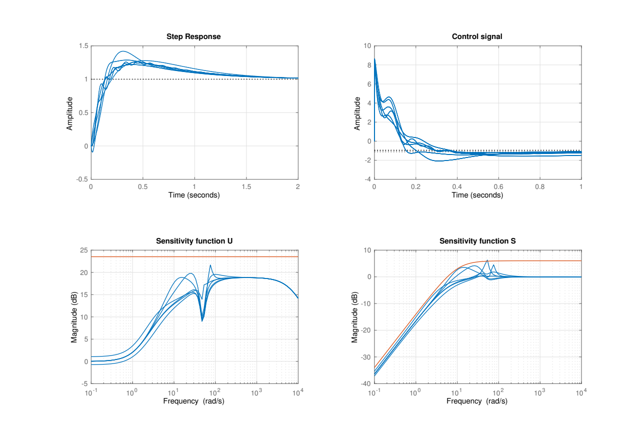
Controller Validation: The bode diagram of the sensitivity function and the step responses for control input and plant output are drawn.



The control signal is too large!

Example (Robust control of a system with multimodel uncertainty)

Controller Redesign: We can add a constraint on the magnitude of $\mathcal{U}(s)$ such that $|\mathcal{U}(j\omega)| < |W_3^{-1}(j\omega)|$ for all ω. For example if we choose $W_3^{-1}(s) = 15$ we have



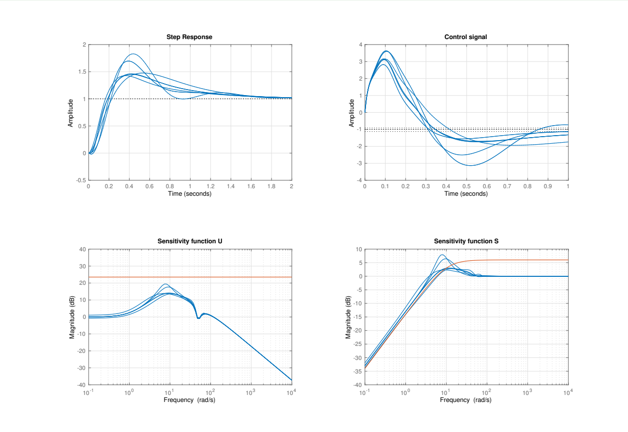

```
W3=1/15; K=mixsyn(Gn,W1,W3,W2);
```

$\mathcal{H}_2$ Control (example from Matlab toolbox)

Example (Robust control of a system with multimodel uncertainty)

$\mathcal{H}_2$ Controller Design: The same problem can be solved by minimizing

$$\|[W_1\mathcal{S} \quad W_2\mathcal{T} \quad W_3\mathcal{U}]\|_2$$




```
P=augw(Gn,W1,W3,W2); K=h2syn(P);
```

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control

State of the art of robust control methods:

- $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control: They minimize the 2- or infinity-norm of one multivariable transfer function (`h2syn, hinfsyn`).
  1. They lead to high order controllers (order of the augmented plant).
  2. We cannot combine 2 and infinity norm for different sensitivity functions or consider the loopshaping performance.
- Other methods have been developed and implemented in Matlab:
  1. Loopshaping with `loopsyn(G,Ld)` that computes a stabilizing controller that minimizes $\|GK - L_d\|_\infty$.
  2. μ-Synthesis approach in which a robust controller is designed for an uncertain model (no need to convert parametric uncertainty to frequency-domain uncertainty). Like the $\mathcal{H}_\infty$ approach $\|T_{zw}\|_\infty$ is minimized by a non-convex optimization algorithm (`dksyn`).
  3. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ with pole placement constraints (`hinfmix`).
  4. Fixed-structure $\mathcal{H}_\infty$ control using `hinfstruc` command that designs low-order controllers by non-smooth optimization algorithms.

Data-Driven Control

Main properties of data-driven method:

- Only the frequency-response data of the system is required (no need to a parametric model).
- Fixed-structure (low-order, centralized, decentralized or distributed) controller is designed by convex optimization.
- Pure time delay (transportation delay or communication delay) is considered in the design.
- Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control can be considered for any sensitivity function or for open-loop shaping.
- Multimodel uncertainty can be directly taken into account.
- It can be used for designing discrete- and continuous-time controllers in the same framework.
- The method needs an initial stabilizing controller.
- The number of frequency data and the frequency range of interest for controller design should be chosen with care.

Data-Driven Control

Plant

Consider an LTI-MIMO system with n_u inputs and n_y outputs and its frequency response $G(e^{j\omega}) \in \mathbb{C}^{n_y \times n_u}$ that can be identified from n_u sets of input/output sampled data as:

$$G(e^{j\omega}) = \left[\sum_{k=0}^{N-1} \mathbb{Y}(k) e^{-j\omega T_s k}\right] \left[\sum_{k=0}^{N-1} \mathbb{U}(k) e^{-j\omega T_s k}\right]^{-1}$$

where N is the number of data points for each experiment. Each column of $\mathbb{U}(k) \in \mathbb{R}^{n_u \times n_u}$ and $\mathbb{Y}(k) \in \mathbb{R}^{n_y \times n_u}$ represents respectively the inputs and the outputs at sample k from one experiment and T_s is the sampling period. Therefore:

$$\omega \in \Omega = \left\{ \omega \,\middle|\, -\frac{\pi}{T_s} \leq \omega \leq \frac{\pi}{T_s} \right\}$$

For simplicity, we assume that $G(e^{j\omega})$ is bounded for all $\omega \in \Omega$. This assumption can be relaxed to include systems with poles on the unit circle.

Data-Driven Control

Controller

A fixed-structure controller is defined as $K = XY^{-1}$ where X and Y are rational stable matrix transfer functions with bounded infinity norm. X with dimension $n_u \times n_y$ and Y with dimension $n_y \times n_y$ are affine in the controller parameters (optimization variables).

Example

Fixed degree controller:
Polynomial controller

$$X = \sum_{k=0}^{n} X_k \cdot \frac{z^k}{(z-\alpha)^n}, \qquad |\alpha| < 1$$

$$Y = \sum_{k=0}^{n} Y_k \cdot \frac{z^k}{(z-\alpha)^n}, \quad Y_k \text{ diagonal}$$

Fixed order controller:
State-space controller

$$X = C_1(zI - A)^{-1}B + D_1$$
$$Y = C_2(zI - A)^{-1}B + D_2$$

A is stable and
(A, B) is controllable.

Data-Driven Control

Centralised, decentralised and distributed fixed degree controller

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \qquad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix}$$

X_{ij}, Y_{ij} are discrete-time stable transfer functions.

Example (PID Controller)

Give a controller parametrisation for designing a SISO PID controller:

$$K(z) = K_p + K_i \frac{z}{z-1} + K_d \frac{z-1}{z} = \frac{K_p z^2 - K_p z + K_i z^2 + K_d z^2 - 2K_d z + K_d}{z(z-1)}$$

We take $X(z) = \dfrac{X_0 + X_1 z + X_2 z^2}{(z-\alpha)^2}$ and $Y(z) = \dfrac{z^2 - z}{(z-\alpha)^2}$.
Note that $Y(z)$ is fixed and $X_0 = K_d, X_1 = -K_p - 2K_d$ and $X_2 = K_p + K_i + K_d$, from which we can compute the PID parameters.

Control Performance

Quadratic Matrix Inequality (QMI)

Consider the application of the Schur Lemma on the following matrix inequalities:

$$\begin{bmatrix} A & B \\ B^* & \Phi^*\Phi \end{bmatrix} \succ 0 \quad \begin{array}{l} \iff \quad \Phi^*\Phi - B^*A^{-1}B \succ 0 \\ \iff \quad A - B(\Phi^*\Phi)^{-1}B^* \succ 0 \end{array}$$

where $A \in \mathbb{C}^{n\times n} \succ 0, B, \Phi \in \mathbb{C}^{n\times n}$ are linear in optimization variables.

QMI Convexification Lemma

The above QMI can be linearized using:

$$(\Phi - \Phi_c)^*(\Phi - \Phi_c) \succeq 0 \quad \Rightarrow \quad \Phi^*\Phi \succeq \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c$$

where Φ_c is any known matrix. Then a sufficient convex condition can be obtained as an LMI:

$$\begin{bmatrix} A & B \\ B^* & \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c \end{bmatrix} \succ 0$$

The conservatism is reduced if Φ_c is close to Φ.

Control Performance

Loop-shaping (∞-norm):

Given a desired open-loop transfer function L_d, compute a controller that minimizes $\|GK - L_d\|_\infty$. This is equivalent to minimizing γ subject to:

$$[G(e^{j\omega})K(e^{j\omega}) - L_d(e^{j\omega})][G(e^{j\omega})K(e^{j\omega}) - L_d(e^{j\omega})]^* \prec \gamma I \qquad \forall\omega \in \Omega$$

Replacing $K(e^{j\omega})$ with $X(e^{j\omega})Y^{-1}(e^{j\omega})$ in the constraint and dropping $(e^{j\omega})$, we obtain:

$$\gamma I - (GX - L_d Y)(Y^* Y)^{-1}(GX - L_d Y)^* \succ 0 \qquad \forall\omega \in \Omega$$

Using QMI convexification Lemma, a convex optimization problem is obtained:

$$\min_{X,Y} \gamma$$

$$\begin{bmatrix} \gamma I & GX - L_d Y \\ (GX - L_d Y)^* & Y^* Y_c + Y_c^* Y - Y_c^* Y_c \end{bmatrix} \succ 0 \qquad \forall\omega \in \Omega$$

where Y_c should be chosen close to Y.

Control Performance

Loop-shaping (2-norm):

In the same way, minimizing $\|GK - L_d\|_2^2$ leads to:

$$\min_K \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} \text{trace}[(GK - L_d)(GK - L_d)^*] d\omega$$

which is equivalent to minimizing the trace of a matrix $\Gamma(\omega) \succ 0$ that satisfies:

$$(GK - L_d)(GK - L_d)^* \prec \Gamma(\omega) \qquad \forall \omega \in \Omega$$

This constraint can be written as: $(GX - L_d Y)(Y^* Y)^{-1}(GX - L_d Y)^* \prec \Gamma(\omega)$. Using QMI convexification Lemma, the following convex optimization problem is obtained:

$$\min_{X,Y} \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} \text{trace}[\Gamma(\omega)] d\omega$$

$$\begin{bmatrix} \Gamma(\omega) & GX - L_d Y \\ (GX - L_d Y)^* & Y^* Y_c + Y_c^* Y - Y_c^* Y_c \end{bmatrix} \succ 0 \qquad \forall \omega \in \Omega$$

Control Performance

$\mathcal{H}_\infty$ Performance

Consider the following $\mathcal{H}_\infty$ problem:

$$\min_K \|W_1 \mathcal{S}\|_\infty \qquad \text{where} \qquad \mathcal{S} = (I + GK)^{-1}$$

Ignoring the closed-loop stability, it can be rewritten as the minimization of γ under a spectral norm constraint:

$$[W_1(I + GK)^{-1}][W_1(I + GK)^{-1}]^* \prec \gamma I \qquad \forall \omega \in \Omega$$

Replacing $K = XY^{-1}$ we obtain: $W_1 Y(Y + GX)^{-1}[(Y + GX)^{-1}]^*(W_1 Y)^* \prec \gamma I$.
Then, taking $\Phi = Y + GX$, we obtain: $W_1 Y(\Phi^*\Phi)^{-1}(W_1 Y)^* \prec \gamma I$.
Choosing $\Phi_c = Y_c + GX_c$, where $K_c = X_c Y_c^{-1}$ is an initial controller, the $\mathcal{H}_\infty$ control can be written as a convex optimization problem:

$$\min_{X,Y} \gamma$$

$$\begin{bmatrix} \gamma I & W_1 Y \\ (W_1 Y)^* & \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c \end{bmatrix} \succ 0 \qquad \forall \omega \in \Omega$$

Control Performance

Mixed Sensitivity Performance Consider the following $\mathcal{H}_\infty$ problem:

$$\min_K \left\| \begin{array}{c} W_1\mathcal{S} \\ W_2\mathcal{U} \end{array} \right\|_\infty \qquad \text{where} \qquad \mathcal{U} = K\mathcal{S} = K(I+GK)^{-1}$$

It can be rewritten as the minimization of γ under a spectral norm constraint:

$$[W_1(I+GK)^{-1}]^*[W_1(I+GK)^{-1}] + [W_2K(I+GK)^{-1}]^*[W_2K(I+GK)^{-1}] \prec \gamma I$$

Replacing $K = XY^{-1}$ and $\Phi = Y + GX$ we obtain:

$$Y^*W_1^*\gamma^{-1}W_1Y + X^*W_2^*\gamma^{-1}W_2X - \Phi^*\Phi \prec 0$$

Using the Schur Lemma we obtain a convex optimization problem:

$$\min_{X,Y} \gamma$$

$$\left[\begin{array}{ccc} \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c & (W_1Y)^* & (W_2X)^* \\ W_1Y & \gamma I & 0 \\ W_2X & 0 & \gamma I \end{array} \right] \succ 0 \qquad \forall \omega \in \Omega$$

Control Performance

$\mathcal{H}_2$ Performance

Consider the following $\mathcal{H}_2$ control problem:

$$\min_K \|W_1\mathcal{S}\|_2^2 \qquad \text{where} \qquad \mathcal{S} = (I + GK)^{-1}$$

Ignoring the closed-loop stability, the integral of the trace of $\Gamma(\omega)$ should be minimized subject to:

$$[W_1(I + GK)^{-1}][W_1(I + GK)^{-1}]^* \prec \Gamma(\omega) \qquad \forall \omega \in \Omega$$

Replacing $K = XY^{-1}$ and $\Phi = Y + GX$ in the constraint, we obtain: $W_1 Y(\Phi^*\Phi)^{-1}(W_1 Y)^* \prec \Gamma(\omega)$. Using QMI convexification Lemma, $\mathcal{H}_2$ control is converted to the following convex optimization problem:

$$\min_{X,Y,\Gamma(\omega)} \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} \text{trace}[\Gamma(\omega)]d\omega$$

$$\begin{bmatrix} \Gamma(\omega) & W_1 Y \\ (W_1 Y)^* & \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c \end{bmatrix} \succ 0, \quad \forall \omega \in \Omega$$

Robust Stability

Multimodel uncertainty

Suppose that the frequency response of a system in m different operating points are available:

$$G(e^{j\omega}) \in \{G_1(e^{j\omega}), G_2(e^{j\omega}), \ldots, G_m(e^{j\omega})\}$$

The robust performance problem for minimzing $\|W_1 S\|_\infty$ is defined as:

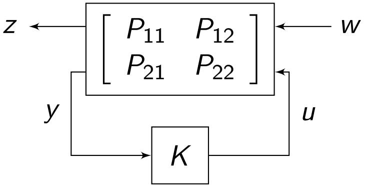
$$\min_{X,Y} \gamma$$

$$\begin{bmatrix} \gamma I & W_1 Y \\ (W_1 Y)^* & \Phi_i^* \Phi_{c_i} + \Phi_{c_i}^* \Phi_i - \Phi_{c_i}^* \Phi_{c_i} \end{bmatrix} \succ 0$$

$$\text{for} \quad i = 1, \ldots, m \quad \text{and} \quad \forall \omega \in \Omega$$

where $\Phi_i = Y + G_i X$ and $\Phi_{c_i} = Y_c + G_i X_c$.

LFT Framework



$$T_{zw} = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

Assumptions:
A1: $P_{21}(j\omega)$ has full row rank $\forall \omega \in \Omega$.
A2: $P(j\omega)$ is bounded $\forall \omega \in \Omega$.

A1 is made to ensure that any possible disturbances have an effect on the measurements. As a result the right inverse of P_{21} exists, i.e. $P_{21}^R = P_{21}^*(P_{21}P_{21}^*)^{-1}$, such that $P_{21}P_{21}^R = I$.
A2 is not fundamental and can be relaxed if there are poles on the imaginary axis.

Let's define $\Phi = P_{21}^R(Y - P_{22}X)$, then its left inverse is $\Phi^L = (Y - P_{22}X)^{-1}P_{21}$. Therefore, denoting $\Psi = I - \Phi\Phi^L = I - P_{21}^R P_{21}$, we have:

$$T_{zw} = P_{11} + P_{12}X\Phi^L = P_{11}(\Phi\Phi^L + \Psi) + P_{12}X\Phi^L = (P_{11}\Phi + P_{12}X)\Phi^L + P_{11}\Psi$$

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control

Subject to closed-loop stability, the synthesis problem can be written as:

$$\min_{K,\Gamma} \gamma$$

$$T_{zw}(j\omega)T_{zw}^*(j\omega) \prec \Gamma(j\omega)$$

for the $\mathcal{H}_\infty$ case, $\Gamma = \gamma I$ and for the $\mathcal{H}_2$ case $\gamma = \int_\Omega trace[\Gamma(j\omega)]d\omega$.

The inequality constraint can be written as:

$$T_{zw}T_{zw}^* = (P_{11}\Phi + P_{12}X)(\Phi^*\Phi)^L(P_{11}\Phi + P_{12}X)^* + (P_{11}\Psi)(P_{11}\Psi)^*$$

using the fact that $\Phi^L\Psi^* = \Phi^L\Psi = \Phi^L - \Phi^L\Phi\Phi^L = 0$.
Then using Shur and QMI Lemma, we obtain the following LMI:

$$\begin{bmatrix} \Gamma - (P_{11}\Psi)(P_{11}\Psi)^* & (P_{11}\Phi + P_{12}X) \\ (P_{11}\Phi + P_{12}X)^* & \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c \end{bmatrix} \succ 0$$

Closed-loop Stability

Theorem

Given a generalized plant model P, the controller $K = XY^{-1}$ stabilizes the closed-loop system if

1. $\det[Y(e^{j\omega})]$ *and* $\det[Y_c(e^{j\omega})]$ *have no zeros on the unit circle.*
2. $\Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c > 0, \quad \forall \omega \in \Omega$
   where $\Phi = P_{21}^R(Y - P_{22}X)$ *and* $\Phi_c = P_{21}^R(Y_c - P_{22}X_c)$ *and*
3. $K_c = X_c Y_c^{-1}$ *is a stabilizing controller.*

- First condition can be relaxed with some infinitely small detours on the Nyquist contour. However there is no need to evaluate $\Phi^*\Phi_c$ on the modified contour because its variation around the zeros of $\det[Y(e^{j\omega})]$ and $\det[Y_c(e^{j\omega})]$ is small and can be ignored.
- The condition $\Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c > 0, \forall \omega \in \Omega$ appears in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control LMIs, so it is always satisfied.
- For the Proof see the course notes.

Implementation Issues

Frequency Gridding

The convex constraints should be satisfied $\forall \omega \in \Omega = [0 \quad \pi/T_s]$ which is a semi-infinite programming. A practical approach is to choose a reasonably large set of frequency samples $\Omega_g = \{\omega_1, \ldots, \omega_g\}$ instead.

Example

The convex optimization problem for loopshaping in 2-norm is as follows:

$$\min_{X,Y} \sum_{k=1}^{g} \text{trace}[\Gamma_k] \Delta\omega_k$$

$$\begin{bmatrix} \Gamma_k & G_k X_k - L_{d_k} Y_k \\ (G_k X_k - L_{d_k} Y_k)^* & Y_k^* Y_{c_k} + Y_{c_k}^* Y_k - Y_{c_k}^* Y_{c_k} \end{bmatrix} \succ 0$$

$$\Phi_k^* \Phi_{c_k} + \Phi_{c_k}^* \Phi_k \succ \Phi_{c_k}^* \Phi_{c_k} \qquad \text{for } k = 1, \ldots, g$$

where the subscript k denotes the frequency response at ω_k, e.g. $G_k = G(e^{j\omega_k})$.

Implementation Issues

Initial controller

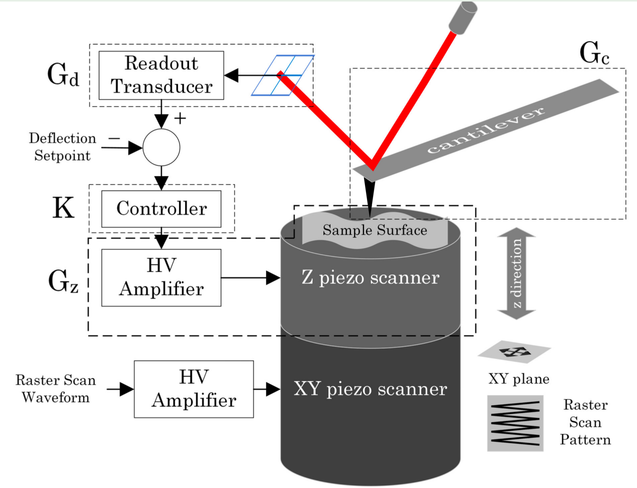
- For stable systems a very low gain controller will always be stabilizing.
- For unstable system, we should have a stabilizing controller for data acquisition in a data-driven setting.
- It can be shown that starting with any initial stabilizing controller, the objective will converge to the global optimal solution when the controller order increases.

Iterative algorithm

- For fixed-structure controllers, the results depend on the initial controller. In fact, we choose a convex set around an initial controller and we find the suboptimal controller in this convex set.
- The results can be improved if the obtained controller is used as an initial controller in a second iteration. It can be shown that the solution will converge to a local optimum of the initial non-convex problem when the number of iterations goes to infinity. In practice after a few iteration the result converges to the vicinity of the optimal solution.

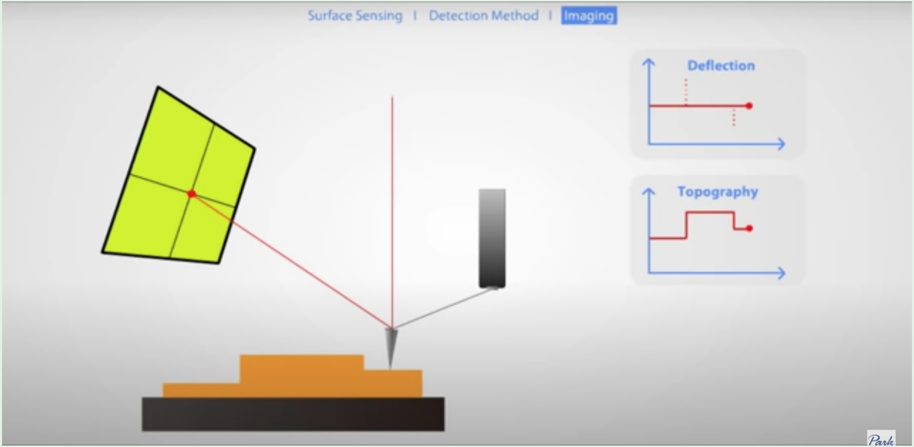
Example (Control of an Atomic Force Microscope)

Working Principle



Example (Control of an Atomic Force Microscope)

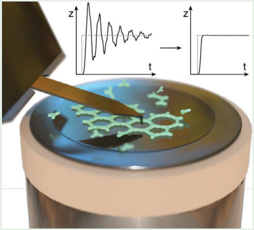
Working Principle

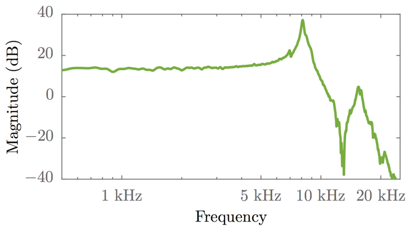


Data-Driven Controller Design

Example (Control of an Atomic Force Microscope)

Frequency-domain data: The input is the vertical position of the sample (100 periods of a PRBS) and the output is the deflection of the cantilever. The sampling frequency is 50 kHz.





Control specifications:

- Closed-loop bandwidth of $\omega_b = 8kHz$.
- No steady-state error and overshoot of less than 5% for step reference.
- A modulus margin of at least 0.5.
- The control signal is limited to 10 v, i.e. $|u(t)| < 10$.

Data-Driven Controller Design

Example (Control of an Atomic Force Microscope)

Design method: Loopshaping with constraints in weighted sensitivity functions

$$\min_K \|GK - L_d\|_2$$

$$\|W_1\mathcal{S}\|_\infty < 1 \qquad \|W_2\mathcal{T}\|_\infty < 1 \qquad \|W_3\mathcal{U}\|_\infty < 1$$

- W_1 is chosen for a bandwidth of 8kHz and a modulus margin of 0.5.
- W_2 is chosen to have the same bandwidth and limit the overshoot to 5% (W_2^{-1} is a low-pass filter with cutoff frequency of ω_b).
- W_3 =cte. is chosen to limit the maximum of the control signal.
- $L_d = \omega_b/s$ is chosen.
- An integrator is fixed in the controller.
- The initial controller is: $K_c(z) = 0.001$.
- $N = 1000$ logarithmically spaced frequency points from 4kHz to 25kHz.

Example (Control of an Atomic Force Microscope)

Convex optimization problem

$$\min \sum_{k=1}^{g} \Gamma_k$$

$$\begin{bmatrix} \Gamma_k & G_k X_k - L_{d_k} Y_k \\ (G_k X_k - L_{d_k} Y_k)^* & Y_k^* Y_{c_k} + Y_{c_k}^* Y_k - Y_{c_k}^* Y_{c_k} \end{bmatrix} \succ 0$$

$$\begin{bmatrix} 1 & W_{1_k} Y_k \\ (W_{1_k} Y_k)^* & \Phi_k^* \Phi_{c_k} + \Phi_{c_k}^* \Phi_k - \Phi_{c_k}^* \Phi_{c_k} \end{bmatrix} \succ 0$$

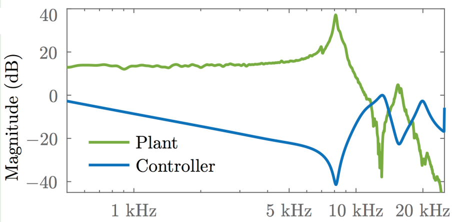
$$\begin{bmatrix} 1 & W_{2_k} G_k X_k \\ (W_{2_k} G_k X_k)^* & \Phi_k^* \Phi_{c_k} + \Phi_{c_k}^* \Phi_k - \Phi_{c_k}^* \Phi_{c_k} \end{bmatrix} \succ 0$$

$$\begin{bmatrix} 1 & W_{3_k} X_k \\ (W_{3_k} X_k)^* & \Phi_k^* \Phi_{c_k} + \Phi_{c_k}^* \Phi_k - \Phi_{c_k}^* \Phi_{c_k} \end{bmatrix} \succ 0$$

$$\text{for} \quad k = 1 \ldots, g$$

10 times faster than a PI controller!

Designed controller



Achieved performance