Stability, Performance and Robustness

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Introduction

Control objective :

The objective in a control system is to make some output y, behave in a desired way by manipulating some input u.

Regulation : Keep y close to some equilibrium point.

Tracking : Keep y close to a reference signal r.

Mathematical model :

In this chapter we consider linear time-invariant models subject to some uncertainty :

$$y = (G + \Delta)u + v$$

v : unknown noise or disturbance.

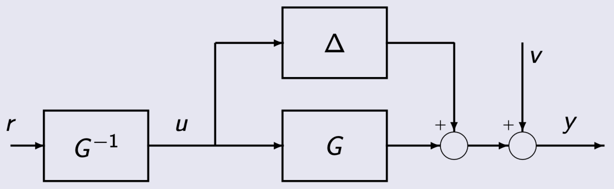
Δ : unknown plant perturbation.

Both v and Δ will be assumed to belong to sets, that is, some a priori information is assumed about v and Δ.

Introduction

Open-loop Solution :

The controller is chosen as the inverse of the plant model.



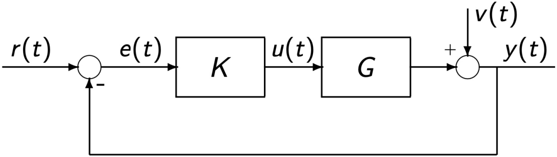
Problems :

- G^{-1} may be not causal or not stable.
- G may be unstable.
- Uncertainty in Δ and v cannot be considered.

Closed-loop Solution :

A feedback controller is designed that guarantees the stability and performance in the presence of uncertainty in Δ and v.

Question



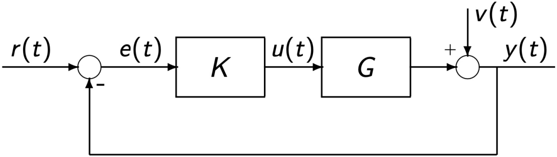
- What is the transfer function between r and y

$$a) \quad \frac{1}{1+GK} \qquad b) \quad \frac{G}{1+GK} \qquad c) \quad \frac{K}{1+GK} \qquad d) \quad \frac{GK}{1+GK}$$

- What is the transfer function between v and y

$$a) \quad \frac{1}{1+GK} \qquad b) \quad \frac{G}{1+GK} \qquad c) \quad \frac{K}{1+GK} \qquad d) \quad \frac{GK}{1+GK}$$

The aim of feedback is to overcome the model uncertainty



Whatever the plant model is, large GK leads to

$$\mathcal{T} = \frac{GK}{1+GK} \approx 1 \,(\text{good tracking}) \quad ; \quad \mathcal{S} = \frac{1}{1+GK} \approx 0 \,(\text{good regulation})$$

For an open-loop stable system :

$K = 0$ (Robust stability) ⟶ $K \to \infty$ (good performance)

No performance — No Robustness

Feedback controller design is a trade-off between robust stability and good performance.

Stability

Stability :

An LTI system represented by a transfer function $G(s)$ is stable if it is analytic in the closed Right Half Plane RHP ($\text{Re } s \geq 0$). In other words, the system is stable if all poles of $G(s)$ are strictly in the Left Half Plane (LHP).

Minimum Phase :

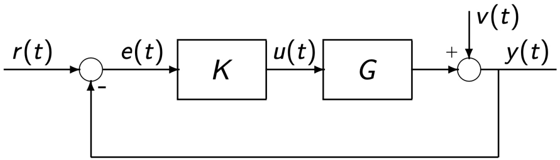
An LTI system represented by a transfer function $G(s)$ is minimum phase if its inverse is stable.

Internal Stability :

A closed-loop system is internally stable if the transfer functions from all external inputs to all internal signals are stable. For a unity feedback system the following four transfer functions should be stable.

$$\frac{1}{1+GK} \qquad \frac{G}{1+GK} \qquad \frac{K}{1+GK} \qquad \frac{GK}{1+GK}$$

Question



Suppose that

$$G(s) = \frac{2s-1}{s^2+2}$$

- Is the closed loop system internally stable with $K(s) = 1$?

  (A) Yes **(B)** No

- Is the closed-loop system internally stable with

$$K(s) = \frac{4s+2}{2s-1}$$

  (A) Yes **(B)** No

Internal Stability

Theorem

A unity feedback system is internally stable if and only if

- *there are no zeros in Re* $s \geq 0$ *in the characteristic polynomial*

$$N_G N_K + M_G M_K = 0$$

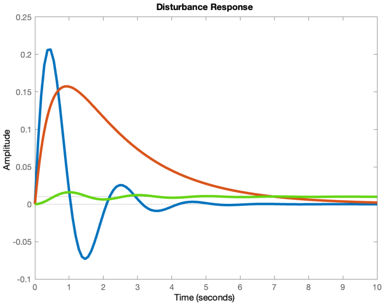
  where

$$G = \frac{N_G}{M_G}, \quad K = \frac{N_K}{M_K}$$

- *or the following two conditions hold :*
  - (a) *The transfer function* $1 + GK$ *has no zeros in Re* $s \geq 0$.
  - (b) *There is no pole-zero cancellation in Re* $s \geq 0$ *when the product* GK *is formed.*
- *or the Nyquist plot of* GK *does not pass through the point -1 and encircles it* n *times counterclockwise, where* n *denotes the number of unstable poles of* G *and* K.

Performance

Consider the following disturbance response for a regulation problem :



- Which one is "smaller" ?
  (A) red **(B)** blue **(C)** green **(D)** It depends !
- Which one is better ?
  (A) red **(B)** blue **(C)** green **(D)** It depends !

Norms for Signals

Consider piecewise continuous signals mapping $(-\infty, +\infty)$ to $\mathbb{R}$. A norm must have the following four properties :

1. $\|u\| \geq 0$ (positivity)
2. $\|au\| = |a| \, \|u\|, \forall a \in \mathbb{R}$ (homogeneity)
3. $\|u\| = 0 \iff u(t) = 0 \quad \forall t$ (positive definiteness)
4. $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality)

1-Norm : $\|u\|_1 = \int_{-\infty}^{\infty} |u(t)| dt$

2-Norm : $\|u\|_2 = \left(\int_{-\infty}^{\infty} u^2(t) dt \right)^{1/2}$

($\|u\|_2^2$ is the total signal energy)

∞-Norm : $\|u\|_\infty = \sup_t |u(t)|$

p-Norm : $\|u\|_p = \left(\int_{-\infty}^{\infty} |u(t)|^p dt \right)^{1/p} \quad 1 \leq p \leq \infty$

Norms for Signals

Example

Show that the one norm has all norm properties :

- Positivity is evident : $\|u\|_1 = \int_{-\infty}^{\infty} |u(t)|dt \geq 0$
- Homogeneity : $\|au\|_1 = \int_{-\infty}^{\infty} |au(t)|dt = |a| \int_{-\infty}^{\infty} |u(t)|dt = |a|\|u\|_1$
- Positive definitness : $\|u\|_1 = \int_{-\infty}^{\infty} |u(t)|dt = 0 \iff u(t) = 0 \quad \forall t$
- Triangle inequality :

$$\begin{aligned}\|u+v\|_1 &= \int_{-\infty}^{\infty} |u(t)+v(t)|dt \\ &\leq \int_{-\infty}^{\infty} |u(t)|dt + \int_{-\infty}^{\infty} |v(t)|dt \\ &\leq \|u\|_1 + \|v\|_1\end{aligned}$$

Question

- Compute the 1-norm, 2-norm and ∞-norm of

$$u(t) = \begin{cases} 2 & 0 \leq t \leq 10 \\ 0 & \text{elsewhere} \end{cases}$$

One-Norm :	**(A)** 10	**(B)** 2	**(C)** 1	**(D)** 20
Two-Norm :	**(A)** 400	**(B)** $2\sqrt{10}$	**(C)** 20	**(D)** 40
∞-Norm :	**(A)** 1	**(B)** 2	**(C)** ∞	**(D)** $\sqrt{2}$

$$\|u\|_1 = \int_0^{10} 2dt = 2t\Big|_0^{10} = 20 \qquad \|u\|_2 = \left(\int_0^{10} 4dt\right)^{1/2} = 2\sqrt{10}$$

- Compute the 1-norm, 2-norm and ∞-norm of $u(t) = \sin \omega t$.

One-Norm :	**(A)** 0	**(B)** 1	**(C)** 2	**(D)**∞
Two-Norm :	**(A)** 0	**(B)** 1	**(C)** 2	**(D)**∞
∞-Norm :	**(A)** 0	**(B)** 1	**(C)** 2	**(D)**∞

- Give a signal with bounded 1-norm and unbounded 2- and ∞-norm.

Norms for Systems (SISO)

Consider linear, time-invariant, causal and finite-dimensional systems.

$$y(t) = g(t) * u(t), \qquad y(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau, \qquad G(s) = \mathcal{L}[g(t)]$$

Properness :

- $G(s)$ is *proper* if $G(j\infty)$ is finite (deg den $\geq$ deg num)
- $G(s)$ is *strictly proper* if $G(j\infty) = 0$ (deg den $>$ deg num)
- $G(s)$ is *biproper* if (deg den $=$ deg num)

Example (Model Reference Control)

Suppose that the objective is to compute a controller such that the closed-loop system $\mathcal{T}(s)$ is close to a reference model $M(s)$. A good controller should make $\|\mathcal{T} - M\|$ small. How can we define a norm for a system ? Which system is smaller ?

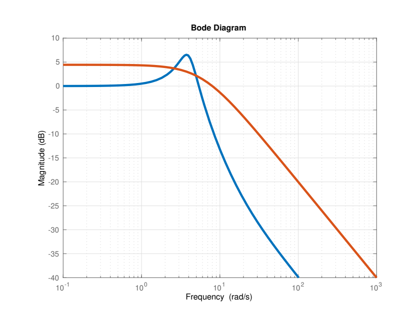
$$E_1(s) = \frac{s+16}{s^2+2s+16} \qquad E_2(s) = \frac{10}{(s+6)}$$

Norms for Systems (SISO)

Frequency response : Let's look at $G(j\omega)$ as a complex infinite dimensional vector. Then similar to the norm for signals, a norm in a vector space can be defined.

Example

Consider the Bode diagram of $E_1(s)$ and $E_2(s)$:



Which one is smaller ?

Norms for Systems (SISO)

2-Norm :

This norm is bounded if $G(s)$ is strictly proper and has no pole on the imaginary axis.

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega\right)^{1/2}$$

Parseval's theorem : Shows the relation between the 2-norm of a system and the 2-norm of its impulse response signal (for stable systems) :

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega\right)^{1/2} = \left(\int_{-\infty}^{\infty} |g(t)|^2 dt\right)^{1/2}$$

∞-Norm :

is bounded if $G(s)$ has no pole on the imaginary axis.

$$\|G\|_\infty = \sup_\omega |G(j\omega)|$$

Norms for Systems (MIMO)

Consider a multi-input multi-output system :

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & G_{13}(s) \\ G_{21}(s) & G_{22}(s) & G_{23}(s) \end{bmatrix}$$

At each ω, $G(j\omega)$ will be a complex matrix.
How can we define the norm of a matrix ?

Norms for matrices :

2-Norm : The spectral norm, induced 2-norm or simply the norm of A is defined as :

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)} = \bar{\sigma}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

F-Norm : *Frobenius norm* is defined as $\|A\|_F = \sqrt{\text{trace}(A^* A)}$, where

$$\text{trace}(A^* A) = \sum_i^n \lambda_i(A^* A) = \sum_i^n \sigma_i^2(A)$$

Norms for Systems (MIMO)

Given $G(s)$ a multi-input multi-output system

2-Norm : This norm is defined as

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\left[G^*(j\omega)G(j\omega)\right] d\omega \right)^{1/2}$$

∞-Norm : This norm is defined as

$$\|G\|_\infty = \sup_\omega \|G(j\omega)\| = \sup_\omega \bar{\sigma}[G(j\omega)]$$

Remarks :

- The two and infinity norm of stable systems are called respectively $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm.
- The infinity norm has an important property (submultiplicative)

$$\|GH\|_\infty \leq \|G\|_\infty \|H\|_\infty$$

Computing the Norms

How to compute the 2-norm :

Suppose that G has bounded two-norm (G is strictly proper), then :

- The two-norm of G can be computed approximately by numerical integration :

$$\|G\|_2^2 \approx \frac{1}{\pi} \sum_{k=0}^{N} |G(j\omega_k)|^2 \Delta\omega_k$$

  where $\Delta\omega_k = \omega_{k+1} - \omega_k$, $\omega_0 = 0$ and N is large enough such that $|G(j\omega_k)|$ can be ignored $\forall k > N$.

- For stable systems, thanks to Parseval's Theorem, the 2-norm can be computed using the impulse response of the system $g(t)$:

$$\|G\|_2^2 = \int_{-\infty}^{\infty} |g(t)|^2 dt \approx \sum_{k=0}^{N} |g(t_k)|^2 \Delta_{t_k}$$

  where N is large enough such that $|g(t_k)|$ can be ignored $\forall k > N$.

Computing the Norms

Example

Compute the 2-norm of $G(s) = \dfrac{1}{\tau s + 1}$, where $\tau > 0$.

Solution : The impulse response is $g(t) = \frac{1}{\tau} e^{-t/\tau}$ for $t \geq 0$, therefore :

$$\|G\|_2^2 = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_0^{\infty} \frac{1}{\tau^2} e^{-2t/\tau} dt = \frac{-\tau}{2\tau^2} e^{-2t/\tau} \Big|_0^{\infty} = \frac{1}{2\tau}$$

For higher order models computing analytically the integral is too difficult. The Residue theorem can be used instead :

Residue Theorem

Consider a complex function $F(z)$ and Γ a closed curve in complex plane that does not pass on any poles of $F(z)$ but encircles n poles $(a_1, \ldots, a_n)$ of $F(z)$, then :

$$\oint_\Gamma F(z) dz = 2\pi j \sum_{k=1}^{n} \text{Res}(F, a_k)$$

where $\text{Res}(F, a_k) = \lim_{z \to a_k} (z - a_k) F(z)$.

Computing the Norms

How to compute the 2-norm using the residue theorem :

By the residue theorem, $\|G\|_2^2$ equals the sum of the residues of $G(-s)G(s)$ at its poles in the left half-plane.

$$
\begin{aligned}
\|G\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G(-s)G(s)ds \\
&= \frac{1}{2\pi j}\oint_{\Gamma} G(-s)G(s)ds \\
&= \sum_{k=1}^{n} \mathrm{Res}[G(-s)G(s), p_k] \qquad \forall p_k \text{ inside } \Gamma
\end{aligned}
$$

- The contour Γ is taken as the imaginary axis and a semicircle with infinite radius that covers the left half plane (LHP).
- The equality comes from the fact that the contribution of the semicircle to the integral equals zero because G is strictly proper.

Computing the Norms

Example

Compute the 2-norm of $G(s) = \frac{K}{s+a}$, where $a > 0$.

(A) $K/\sqrt{a}$ **(B)** 0 **(C)** $K/\sqrt{2a}$ **(D)** $K^2/2a$

Solution : We have

$$G(-s)G(s) = \frac{K}{-s+a}\frac{K}{s+a} = \frac{K^2}{(-s+a)(s+a)}$$

There is only one pole $p_1 = -a$ in LHP and its residue at p_1 is :

$$\|G\|_2^2 = \text{Res}[G(-s)G(s), p_1] = \lim_{s \to -a}(s+a)\frac{K^2}{(-s+a)(s+a)} = \frac{K^2}{2a}$$

$$\Rightarrow \|G\|_2 = \frac{K}{\sqrt{2a}}$$

Computing the Norms

How to compute the ∞-norm :

Choose a fine grid of frequency points $\{\omega_1, \dots, \omega_N\}$, then

$$\text{SISO} : \|G\|_\infty \approx \max_{1 \le k \le N} |G(j\omega_k)| \qquad \text{MIMO} : \|G\|_\infty \approx \max_{1 \le k \le N} \bar{\sigma}[G(j\omega_k)]$$

or alternatively, solve $\frac{d|G(j\omega)|^2}{d\omega} = 0$

Example

Compute the infinity norm of

$$G(s) = \frac{as+1}{bs+1} \qquad a, b > 0$$

If $a \ge b$: **(A)** 1 **(B)** b/a **(C)** a/b **(D)** ∞
If $a < b$: **(A)** 1 **(B)** b/a **(C)** a/b **(D)** ∞

Computing the Norms (state-space methods)

Consider a state-space model for a stable strictly proper system :

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad ; \qquad y(t) = Cx(t)$$

2-Norm :

The $\mathcal{H}_2$ norm of G is given by :

$$\|G\|_2 = \sqrt{\text{trace}[CLC^T]}$$

where $L = L^T \succ 0$ is a symmetric positive definite solution to the following equation :

$$AL + LA^T + BB^T = 0$$

Computing the Norms (state-space methods)

Proof

The impulse response of the system is given by : $g(t) = Ce^{tA}B$ for $t > 0$. Calling on Parseval we get :

$$\|G\|_2^2 = \|g\|_2^2 = \text{trace} \int_0^\infty g(t)g^T(t)dt = \text{trace} \int_0^\infty Ce^{tA}BB^Te^{tA^T}C^Tdt$$

which is equal to trace$[CLC^T]$, where

$$L = \int_0^\infty e^{tA}BB^Te^{tA^T}dt$$

Now, integrate both sides of the following equation :

$$\frac{d}{dt}e^{tA}BB^Te^{tA^T} = Ae^{tA}BB^Te^{tA^T} + e^{tA}BB^Te^{tA^T}A^T$$

from 0 to ∞, to get $-BB^T = AL + LA^T$.

Computing the Norms

Example

Compute the 2-norm of $G(s) = \dfrac{K}{s+a}$, where $a > 0$ using the state-space method.

Solution : We first compute the state-space representation of $G(s)$:

$$A = -a \qquad B = K \qquad C = 1 \qquad D = 0$$

Then we solve the following equation for L :

$$AL + LA^T + BB^T = 0 \quad \Rightarrow \quad -aL - La + K^2 = 0 \quad \Rightarrow \quad L = \frac{K^2}{2a}$$

$$\Rightarrow \quad \|G\|_2^2 = \text{trace}(CLC^T) = L = \frac{K^2}{2a} \quad \Rightarrow \quad \|G\|_2 = \frac{K}{\sqrt{2a}}$$

Computing the Norms (state-space methods)

Lemma (Bounded Real Lemma)

Consider a strictly proper stable LTI system G and $\gamma > 0$. Then $\|G\|_\infty < \gamma$, if and only if the Hamiltonian matrix H has no eigenvalue on the imaginary axis.

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \qquad H = \begin{bmatrix} A & \gamma^{-2}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

Proof : (Sufficiency, SISO case) Consider $\Phi(s) = [1 - \gamma^{-2}G^T(-s)G(s)]$ then it can be shown that H is the state matrix of $\Phi^{-1}(s)$.

- $\|G\|_\infty < \gamma$ if and only if $\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$
- Since $G(s)$ is strictly proper $\Phi(j\infty) = 1 > 0$.
- Since $\Phi(j\omega)$ is a continuous function of ω, $\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$ if and only if $\Phi(j\omega)$ is never equal to zero or $\Phi^{-1}(s)$ has no pole on the imaginary axis.
- Therefore, if H has no eigenvalues on the imaginary axis $\|G\|_\infty < \gamma$.

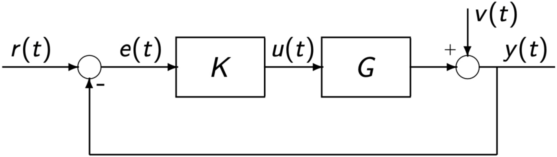
Computing the Norms (state-space methods)

Computing $\mathcal{H}_\infty$ norm : The bounded real lemma and the bisection optimization algorithm can be used to compute the $\mathcal{H}_\infty$ norm of G :

Bisection algorithm :

1. Select an upper bound γ_u and a lower bound γ_l such that $\gamma_l \leq \|G\|_\infty \leq \gamma_u$.
2. If $(\gamma_u - \gamma_l)/\gamma_l <$ specified level, stop ; $\|G\|_\infty \approx (\gamma_u + \gamma_l)/2$. Otherwise go to the next step.
3. Set $\gamma = (\gamma_u + \gamma_l)/2$;
4. Test if $\|G\|_\infty < \gamma$ by calculating the eigenvalues of H for the given γ.
5. If H has an eigenvalue on the imaginary axis, set $\gamma_l = \gamma$, otherwise set $\gamma_u = \gamma$ and go back to step 2.

If we know how big the input is, how big is the output going to be ?



- If $v(t)$ is a step disturbance then what will be the norm of $y(t)$?
- If $v(t) = \sin(\omega t)$ then what will be the norm of $y(t)$?
- if $\|v(t)\|_2 \leq 1$ then what will be the upper bound of $\|y(t)\|_{2,\infty}$?
- if $\|v(t)\|_\infty \leq 1$ then what will be the upper bound of $\|y(t)\|_{2,\infty}$?

Input-output relationships

Known input : Consider an LTI system $G(s)$ with input $u(t)$ and output $y(t)$ and the impulse response $g(t)$, then :

Output Norms for Two Inputs

$u(t)$	$\delta(t)$	$\sin(\omega t)$
$\|y\|_2$	$\|G\|_2$	∞
$\|y\|_\infty$	$\|g\|_\infty$	$\|G(j\omega)\|$

Proofs :

- If $u(t) = \delta(t)$ then $y(t) = g(t)$, therefore :
  - $\|y\|_2 = \|g\|_2 = \|G\|_2$
  - $\|y\|_\infty = \|g\|_\infty$

- If $u(t) = \sin(\omega t)$ then $y(t) = |G(j\omega)| \sin(\omega t + \phi)$, therefore :
  - $\|y\|_2 = \infty$
  - $\|y\|_\infty = |G(j\omega)|$

Input-output relationships

Bounded norm input : Consider an LTI system $G(s)$ with input $u(t)$ and output $y(t)$ and the impulse response $g(t)$, then :

System Gains :

	$\|u\|_2 = 1$	$\|u\|_\infty = 1$
$\|y\|_2$	$\|G\|_\infty$	∞
$\|y\|_\infty$	$\|G\|_2$	$\|g\|_1$

Entry (1,1) : We have

$$\begin{aligned} \|y\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 |U(j\omega)|^2 d\omega \leq \|G\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(j\omega)|^2 d\omega \\ &\leq \|G\|_\infty^2 \|U\|_2^2 = \|G\|_\infty^2 \|u\|_2^2 \end{aligned}$$

Two-norm system gain equals the infinity norm of the system

$$\boxed{\|G\|_\infty = \sup_{u\neq 0} \frac{\|y\|_2}{\|u\|_2}}$$

Bounded norm input :

System Gains :

	$\|u\|_2 = 1$	$\|u\|_\infty = 1$
$\|y\|_2$	$\|G\|_\infty$	∞
$\|y\|_\infty$	$\|G\|_2$	$\|g\|_1$

Entry(2,1) : According to the Cauchy-Schwartz inequality

$$|y(t)| = \left|\int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau\right| \leq \left(\int_{-\infty}^{\infty} g^2(t-\tau)d\tau\right)^{1/2} \left(\int_{-\infty}^{\infty} u^2(\tau)d\tau\right)^{1/2}$$

$$= \|g\|_2\|u\|_2 = \|G\|_2\|u\|_2 \Rightarrow \|y\|_\infty \leq \|G\|_2\|u\|_2$$

Entry (2,2) : We have

$$\begin{aligned} |y(t)| &= \left|\int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau\right| \leq \int_{-\infty}^{\infty} |g(t-\tau)||u(\tau)|d\tau \\ &\leq \|u\|_\infty \int_{-\infty}^{\infty} |g(t-\tau)|d\tau = \|g\|_1\|u\|_\infty \end{aligned}$$

Asymptotic Tracking

Internal Model Principle : For perfect asymptotic tracking of $r(t)$, the loop transfer function $L = GK$ must contain the unstable poles of $r(s)$.

Theorem

Assume that the feedback system is internally stable and n=d=0.

(a) If $r(t)$ is a step, then $\lim_{t\to\infty} e(t) = r(t) - y(t) = 0$ iff $\mathcal{S} = (1+L)^{-1}$ has at least one zero at the origin.

(b) If $r(t)$ is a ramp, then $\lim_{t\to\infty} e(t) = 0$ iff $\mathcal{S}$ has at least two zeros at the origin.

(c) If $r(t) = \sin(\omega t)$, then $\lim_{t\to\infty} e(t) = 0$ iff $\mathcal{S}$ has at least one zero at $s = j\omega$.

Final-Value Theorem :

If $y(s)$ has no poles in Re $s \geq 0$ except possibly one pole at $s = 0$ then :

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s)$$

Asymptotic Tracking

Proof (a) :

$$r(s) = \frac{c}{s} \text{ and } e(s) = \mathcal{S}(s)\frac{c}{s} \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\mathcal{S}(s)\frac{c}{s}$$

The limit is zero iff $\mathcal{S}$ has at least one zero at origin. For this, GK should have a pole at origin, because : $\mathcal{S}(s) = \frac{1}{1+GK} = \frac{1}{1+\frac{N_G N_K}{D_G D_K}} = \frac{D_G D_K}{D_G D_K + N_G N_K}$

Proof (b) :

$$r(s) = \frac{c}{s^2} \text{ and } e(s) = \mathcal{S}(s)\frac{c}{s^2} \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\mathcal{S}(s)\frac{c}{s^2}$$

The limit is zero iff $\mathcal{S}$ has at least two zeros at origin (or GK two poles at origin).

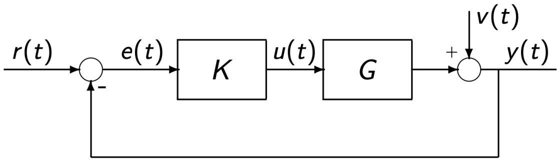
Proof (c) :

$$r(s) = \frac{c}{s^2+\omega_0^2} \text{ and } e(s) = \mathcal{S}(s)\frac{c}{s^2+\omega_0^2} \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\mathcal{S}(s)\frac{c}{s^2+\omega_0^2}$$

The limit is zero iff $\mathcal{S}$ has at least one zero at $j\omega_0$ (the other will be at $-j\omega$). For this, GK should have two poles at $\pm j\omega_0$.

Question

Consider the following closed-loop system :

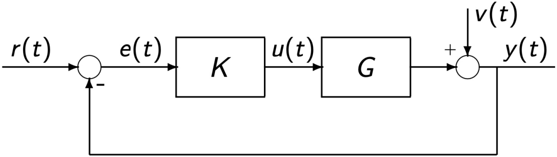


Which criterion should be minimized to minimize the two norm of the input when $r(t) = 0$ and $v(t)$ is a Dirac impulse signal.

(A) $\left\| \frac{GK}{1+GK} \right\|_\infty$ **(B)** $\left\| \frac{K}{1+GK} \right\|_2$ **(C)** $\left\| \frac{K}{1+GK} \right\|_\infty$ **(D)** $\left\| \frac{1}{1+GK} \right\|_2$

Question

Consider the following closed-loop system :

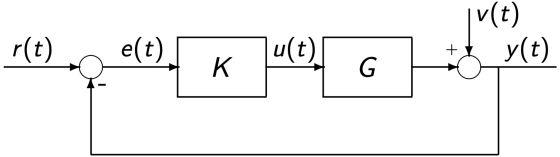


Which criterion should be minimized to minimize the two norm of the tracking error when $v(t) = 0$ and $r(t)$ is a step signal.

(A) $\left\| \frac{1}{1+GK} \right\|_2$ **(B)** $\left\| \frac{1/s}{1+GK} \right\|_2$ **(C)** $\left\| \frac{1}{1+GK} \right\|_\infty$ **(D)** $\left\| \frac{1/s}{1+GK} \right\|_\infty$

Question

Consider the following closed-loop system :



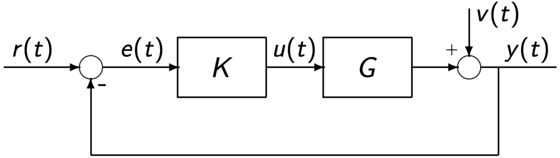
Which criterion should be minimized to minimize the two norm of the output when $r(t) = 0$ and $v(t) = \sin \omega_0 t$.

(A) $\left\| \frac{(s^2 + \omega_0^2)^{-1}}{1 + GK} \right\|_2$ **(B)** $\left\| \frac{\omega_0^2}{1 + GK} \right\|_2$

(C) $\left\| \frac{(s^2 + \omega_0^2)^{-1}}{1 + GK} \right\|_\infty$ **(D)** $\left\| \frac{(s^2 + \omega_0^2)}{1 + GK} \right\|_\infty$

Question

Consider the following closed-loop system :



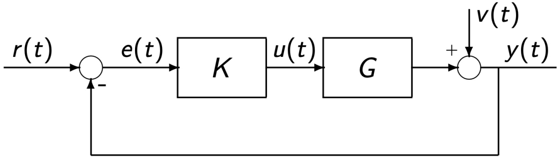
- Which criterion should be minimized to minimize the infinity norm of the tracking error when $v(t) = 0$ and $r(t) = \sin \omega_0 t$.

(A) $\left\| \frac{\mathcal{S}}{(s^2+\omega_0^2)} \right\|_2$ **(B)** $|\mathcal{S}(j\omega_0)|$ **(C)** $\left\| \frac{\mathcal{S}}{(s^2+\omega_0^2)} \right\|_\infty$ **(D)** $\|\mathcal{S}\|_\infty$

- What about if $r(t) = \sin \omega t$ and $\omega_1 \leq \omega \leq \omega_2$?

Question

Consider the following closed-loop system :



- Which criterion should be minimized to minimize the two norm of the output when $r(t) = 0$ and $v(t)$ is a bounded two-norm signal.

(A) $\left\| \frac{1}{1+GK} \right\|_2$ **(B)** $\left\| \frac{1/s}{1+GK} \right\|_2$ **(C)** $\left\| \frac{1}{1+GK} \right\|_\infty$ **(D)** $\left\| \frac{1/s}{1+GK} \right\|_\infty$

- What about if the energy of $v(t)$ is concentrated between ω_1 and ω_2 ?

Nominal Performance

Performance Specification :

Many performance specifications can be represented by minimization of a weighted closed-loop transfer function. Typically, the following criterion is considered in this course :

$$\boxed{\min_K \|W_1\mathcal{S}\|}$$

- $W_1(s)$ is called the **performance filter** and typically is a low-pass filter.
- If the external signal (i. e. $r(t)$ or $v(t)$) is known (e.g. step, ramp, sinusoid, etc), the 2-norm is minimized. In this case, a good choice for $W_1(s)$ is the Laplace transform of the external signal.
- If the external signal belongs to the set of bounded 2-norm signals, the ∞-norm is minimized. In this case, a good choice for $W_1(s)$ is an upper bound on the spectrum of the signals in the set.
- Depending on the application, other sensitivity functions can also be considered.

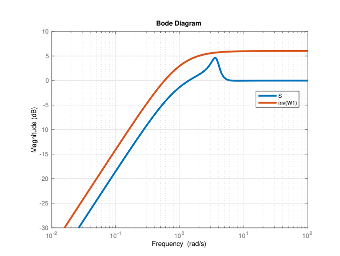
Nominal Performance

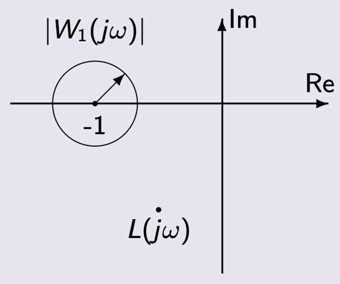
In many applications, the nominal performance can be defined as a constraint :

$$|\mathcal{S}(j\omega)| < |W_1^{-1}(j\omega)| \quad \forall\omega \quad \Rightarrow \quad |W_1(j\omega)\mathcal{S}(j\omega)| < 1 \quad \forall\omega \quad \Rightarrow \quad \boxed{\|W_1\mathcal{S}\|_\infty < 1}$$

where $W_1(s)$ is typically a low-pass filter.

Graphical interpretation :





$$\left|\frac{W_1(j\omega)}{1+L(j\omega)}\right| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |W_1(j\omega)| < |1+L(j\omega)| \quad \forall\omega$$

The open-loop transfer function $L(j\omega)$ should not intersect the performance disk.

Model Uncertainty

Model Uncertainty : Physical systems cannot be exactly modelled. They belong to an *uncertainty model set*, which can be *structured* or *unstructured*.

Structured model set

Parametric uncertainty :

$$\mathcal{G} = \left\{ \frac{K}{\tau s + 1} : \tau_{\text{min}} \leq \tau \leq \tau_{\text{max}}, K_{\text{min}} \leq K \leq K_{\text{max}} \right\}$$

Multimodel uncertainty : $\mathcal{G} = \{G_0, G_1, G_2, G_3\}$

Unstructured model set

Norm bounded uncertainty : $\mathcal{G} = \{G_0 + \Delta : \quad \|\Delta\|_\infty \leq \gamma\}$

Frequency-domain uncertainty :

$$\mathcal{G} = \{G(j\omega) | |S_1(j\omega)| < |G(j\omega)| < |S_2(j\omega)|\}$$

Model Uncertainty

Example (Norm bounded uncertainty)

Consider a plant model with unmodelled dynamics :

$$\tilde{G}(s) = \frac{12}{(s+2)(s+3)}\frac{1}{0.1s+1}$$

where $\tilde{G}(s)$ is the true model. The objective is to find a norm bounded uncertainty set for this model as

$$\tilde{G} \in \{G + \Delta : \quad \|\Delta\|_\infty \leq \gamma\}$$

Solution : Let's write $\tilde{G}(s)$ as :

$$\tilde{G}(s) = \frac{15}{(s+2)} + \frac{-12/0.7}{(s+3)} + \frac{15/7}{s+10} = \underbrace{\frac{-2.143s+10.71}{s^2+5s+6}}_{G(s)} + \underbrace{\frac{15/7}{s+10}}_{\Delta}$$

It is clear that $\gamma = \|\Delta\|_\infty = 15/70 = 0.214$

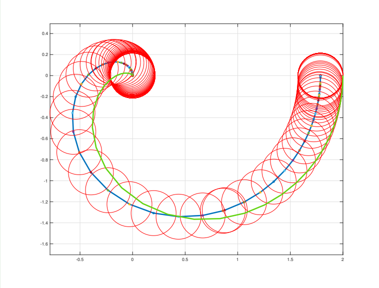
Model Uncertainty

Example (Norm bounded uncertainty)

The uncertainty model set can be presented in the Nyquist diagram :

$$\tilde{G} \in \{G + \Delta : \quad \|\Delta\|_\infty \leq \gamma\}$$

- Blue : Nominal Model G
- Green : True model $\tilde{G}$
- Red : Uncertainty set $G + \Delta$



Can we reduce the size of the uncertainty set ?

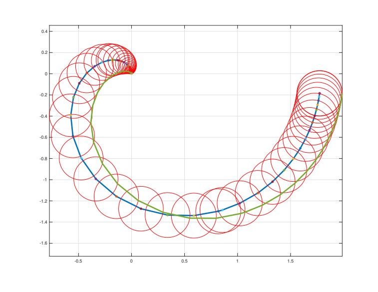
Model Uncertainty

Example (Norm bounded uncertainty)

$$\tilde{G}(s) = \frac{12}{(s+2)(s+3)}\frac{1}{0.1s+1} = \underbrace{\frac{-2.143s+10.71}{s^2+5s+6}}_{G(s)} + \underbrace{\frac{15/7}{s+10}}_{W_2\Delta}$$

- Blue : Nominal Model G
- Green : True model $\tilde{G}$
- Red : Uncertainty set $G + W_2\Delta$

$$W_2(s) = \frac{15/7}{s+10} \quad , \quad \|\Delta\|_\infty \leq 1$$

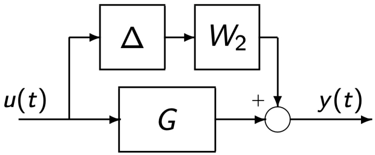


The radius of the uncertainty disk at each frequency is $|W_2(j\omega)|$, which presents the size of uncertainty.

Unstructured uncertainty

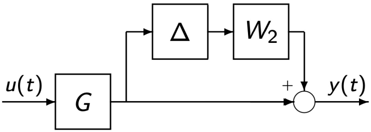
Additive uncertainty :

$$\tilde{G} = G + \Delta W_2 \qquad \|\Delta\|_\infty \leq 1$$



Multiplicative uncertainty :

$$\tilde{G} = G(1 + \Delta W_2) \qquad \|\Delta\|_\infty \leq 1$$



$\tilde{G}$: true model $\qquad$ G : nominal model

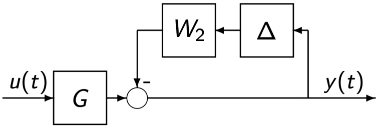
Δ : norm-bounded uncertainty $\qquad$ W_2 : Stable weighting filter

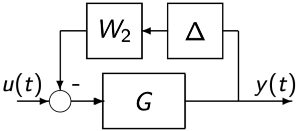
Remark : It is assumed that G and $\tilde{G}$ have the same number of unstable poles.

Multiplicative uncertainty can be converted to additive uncertainty by changing the weighting filter : $W_2^{\text{add}} \equiv GW_2^{\text{mul}}$

Unstructured uncertainty

Feedback uncertainty :





$$\tilde{G} = \frac{G}{1 + \Delta W_2} \qquad \|\Delta\|_\infty \leq 1$$

$$\tilde{G} = \frac{G}{1 + \Delta W_2 G} \qquad \|\Delta\|_\infty \leq 1$$

$\tilde{G}$: true model

G : nominal model

Δ : norm-bounded uncertainty

W_2 : Stable weighting filter

Remark : It is assumed that G and $\tilde{G}$ have the same number of unstable poles.

All unstructured uncertainty models are equivalent from a theoretical point of view, however, one may be preferred for some applications because the computation of the weighting filter becomes simpler.

Model Uncertainty

Remarks :

- There are specific methods for analysis and control synthesis of systems with structured (multimodel or parametric) uncertainty and unstructured (frequency-domain) uncertainty.
- Structured uncertainty can be converted to unstructured uncertainty.
- If we can analyze and synthesize closed-loop systems with unstructured uncertainty, we can find a solution to many robust control problems.
- Controller design for a model set greater than the real model set leads to a *conservative* design.

Model Uncertainty

Converting structured to unstructured uncertainty

Multimodel to multiplicative uncertainty

Problem : A multimodel uncertainty set $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$ is given. Find the uncertainty filter $W_2(s)$ in the multiplicative uncertainty set $\tilde{G} = G(1 + \Delta W_2)$.

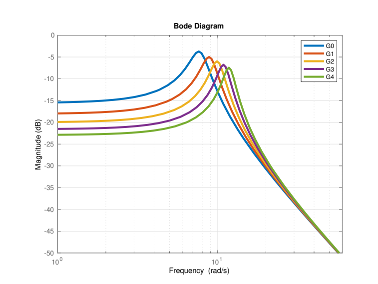
- Choose one of the models as the nominal model G. Then we have :

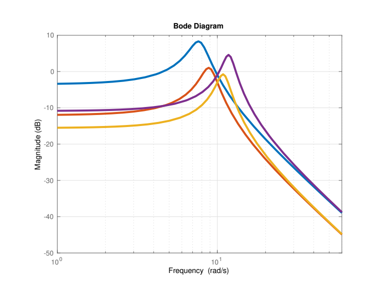
$$\tilde{G} = G(1 + \Delta W_2) \Rightarrow \frac{G_i}{G} - 1 = \Delta W_2 \qquad \text{for } i = 1, \ldots, m$$

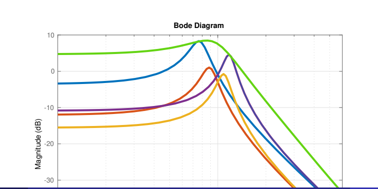
- Since $\|\Delta\|_\infty \leq 1 \quad \Rightarrow \left|\frac{G_i(j\omega)}{G(j\omega)} - 1\right| \leq |W_2(j\omega)| \qquad \text{for } i = 1, \ldots, m$
- Compute $\overline{W_2}(j\omega)$ such that $|\overline{W_2}(j\omega)| = \max_i \left|\frac{G_i(j\omega)}{G(j\omega)} - 1\right| \qquad \forall\omega$
- Design $W_2(s)$ such that $|W_2(j\omega)| \geq |\overline{W_2}(j\omega)| \quad \forall\omega.$

Example (Multimodel to multiplicative uncertainty)

Suppose that $\mathcal{G} = \{G_0, G_1, G_2, G_3, G_4\}$ is given. Compute a 3rd order uncertainty filter for the multiplicative uncertainty set.







Model Uncertainty

Example (Parametric to multiplicative uncertainty)

Given $\tilde{G}(s) = \left\{ \frac{k}{s-2} : 0.1 \leq k \leq 10 \right\}$ compute the uncertainty filter $W_2(s)$.

- First we choose a nominal model : $G(s) = \frac{k_0}{s-2}$ with $k_0 = 5.05$
- Then we compute :

$$\left| \frac{\tilde{G}(j\omega)}{G(j\omega)} - 1 \right| \leq |W_2(j\omega)| \Rightarrow \max_{0.1 \leq k \leq 10} \left| \frac{k}{5.05} - 1 \right| \leq |W_2(j\omega)|$$

- By inspection we obtain $W_2(s) = 4.95/5.05 = 0.98$.

- Similar solution could be obtained by sampling k in the interval $0.1 \leq k \leq 10$ and converting the multimodel to multiplicative uncertainty.
- What is the uncertainty filter for additive uncertainty set ?

(A) $\frac{0.98(s-2)}{5.05}$ **(B)** $\frac{4.95}{s-2}$ **(C)** $\frac{4.95}{s+2}$ **(D)** $\frac{0.98}{s-2}$

Model Uncertainty

Example (Parametric to feedback uncertainty)

Given $\tilde{G}(s) = \left\{ \frac{1}{s^2 + as + 1} : 0.4 \leq a \leq 0.8 \right\}$ compute the uncertainty filter $W_2(s)$ in a feedback uncertainty set.

- Choose the nominal model as $G(s) = \frac{1}{s^2 + 0.6s + 1}$
- Represent the uncertain parameter as a function of Δ :

$$a = 0.6 + 0.2\Delta, \quad -1 \leq \Delta \leq 1$$

- The uncertainty set is given by :

$$\tilde{G}(s) = \frac{1}{s^2 + 0.6s + 0.2\Delta s + 1} = \frac{\frac{1}{s^2+0.6s+1}}{1 + \frac{0.2\Delta s}{s^2+0.6s+1}} = \frac{G(s)}{1 + \Delta W_2(s) G(s)}$$

- This gives $W_2(s) = 0.2s$.

Model Uncertainty

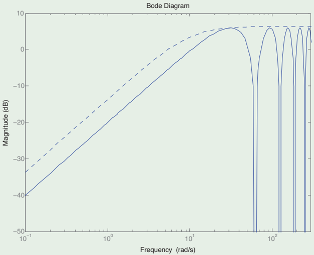
Example (Time-delay to multiplicative uncertainty)

Given $\tilde{G}(s) = e^{-\tau s}\dfrac{1}{s^2}$ where $0 \le \tau \le 0.1$, compute $W_2(s)$ in a multiplicative uncertainty set with the nominal model as $G(s) = \dfrac{1}{s^2}$.

- For multiplicative uncertainty we should have :

$$\left|\frac{\tilde{G}(j\omega)}{G(j\omega)} - 1\right| \le |W_2(j\omega)| \Rightarrow |e^{-\tau j\omega} - 1| \le |W_2(j\omega)| \quad \forall \omega, \tau$$

- The worst case happens for $\tau = 0.1$.
- The Bode diagram of $|e^{-0.1j\omega} - 1|$ is given.
- Using the Bode diagram we can find $W_2(s) = \dfrac{0.21s}{0.1s+1}$.



Model Uncertainty

Stochastic uncertainty

- Different models in an uncertainty model set may have different probabilities.
- Large deterministic uncertainties lead to robust controllers with low performance.
- Stochastic uncertainty model sets may reduce the conservatism and lead to high performance controllers.
- Identification methods lead to nonparametric and parametric models with stochastic uncertainty (because of measurement noise).
- For stochastic uncertainty model sets we cannot guarantee the closed-loop stability in a deterministic sense.

Stochastic Uncertainty

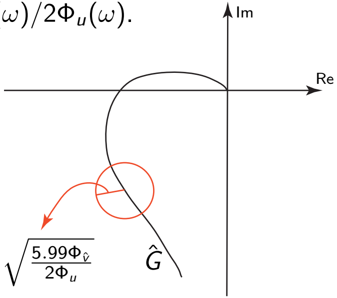
Nonparametric uncertainty : The frequency-domain model of a system can be estimated by the Fourier transform method :

$$\hat{G}(e^{j\omega}) = \frac{Y(\omega)}{U(\omega)} + \frac{V(\omega)}{U(\omega)} = G(e^{j\omega}) + \frac{V(\omega)}{U(\omega)}$$

- The variance of $\hat{G}$ is $\Phi_v(\omega)/\Phi_u(\omega)$.
- The estimates $R_e\{\hat{G}(e^{j\omega})\}$ and $I_m\{\hat{G}(e^{j\omega})\}$ are asymptotically uncorrelated and normally distributed with a variance of $\Phi_v(\omega)/2\Phi_u(\omega)$.
- Therefore, $|\hat{G}|^2$ has a chi-squared distribution (or $|\hat{G}|$ has a Rayleigh distribution).

Knowing the distribution of $|\hat{G}|$, the 0.95% confidence interval in the Nyquist diagram can be computed.

$$G = \hat{G} + W_3(\omega)\Delta$$



$$W_3 = \sqrt{\frac{5.99\Phi_{\hat{v}}}{2\Phi_u}}$$

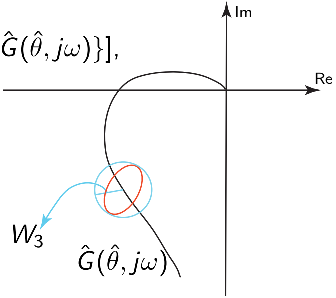
Stochastic Uncertainty

Parametric uncertainty : The parametric model of a system can be estimated using the prediction error method. The covariance of the parameters can also be estimated based on the data.

- If $\hat{\theta}$ is a random variable with Gaussian distribution $\mathcal{N}(\theta_0, P)$, then $f(\hat{\theta})$ converge in distribution to a normal distribution :

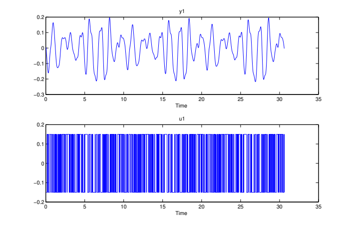
$$\mathcal{N}\left(f(\theta_0), \left(\frac{\partial f}{\partial \theta}\right) P \left(\frac{\partial f}{\partial \theta}\right)^T\right)$$

- Therefore, if we take $f(\hat{\theta}) = [R_e\{\hat{G}(\hat{\theta}, j\omega)\} \ I_m\{\hat{G}(\hat{\theta}, j\omega)\}]$, we can compute the covariance of f and the uncertainty in the Nyquist diagram.

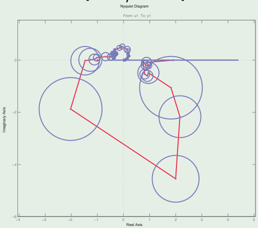


Since P is not diagonal, the frequency-domain uncertainty will be an ellipse.
So W_3 is the radius of the smallest disk that covers the ellipse.

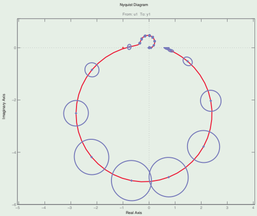
Example



Input/output data of an electromechanical system



Uncertainty from nonparametric identification



Uncertainty from parametric identification

Robust Stability

Definition

Robustness : A controller is robust with respect to a closed-loop characteristic, if this characteristic holds for every plant in $\mathcal{G}$.

Definition

Robust Stability : A controller is robust in stability if it provides internal stability for every plant in $\mathcal{G}$.

Definition

Stability margin : For a given model set with an associate size, it can be defined as the largest model set stabilized by a controller.

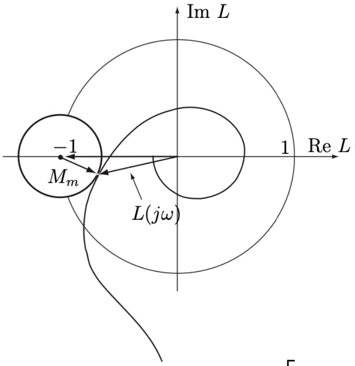
Definition

Stability margin for an uncertainty model : Given $\tilde{G} = G(1 + \Delta W_2)$ with $\|\Delta\|_\infty \leq \beta$, the stability margin for a controller C is the least upper bound of β.

Robust Stability

Definition

Modulus margin : The shortest distance from -1 to the open-loop Nyquist curve.



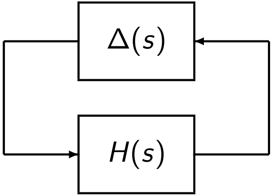
$$M_m = \inf_{\omega} |-1 - L(j\omega)| = \inf_{\omega} |1 + L(j\omega)| = \left[\sup_{\omega} \frac{1}{1 + L(j\omega)}\right]^{-1} = \|\mathcal{S}\|_{\infty}^{-1}$$

Robust Stability

Theorem (Small Gain)

Suppose H is stable and has bounded infinity norm and let $\gamma > 0$. The following feedback loop is internally stable for all stable $\Delta(s)$ with

$$\|\Delta\|_\infty \leq 1/\gamma \quad \textit{if and only if} \quad \|H\|_\infty < \gamma$$

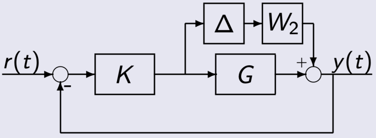


Robust stability condition for plants with additive uncertainty :

$$\tilde{G} = G + \Delta W_2 \Rightarrow H = W_2 \frac{-K}{1 + GK}$$

Closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ iff

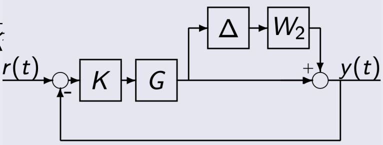
$$\|W_2 K \mathcal{S}\|_\infty < 1$$



Robust Stability

Robust stability condition for plants with multiplicative uncertainty :

$$\tilde{G} = G(1+\Delta W_2) \Rightarrow H = W_2 \frac{-GK}{1+GK}$$



Closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ *iff* $\|W_2\mathcal{T}\|_\infty < 1$.

Proof : Assume that $\|W_2\mathcal{T}\|_\infty < 1$. We show that the winding number of $1+GK$ around zero is equal to that of $1+\tilde{G}K$.

$$1+\tilde{G}K = 1+GK(1+\Delta W_2) = 1+GK+GK\Delta W_2 = 1+GK+(1+GK)\mathcal{T}\Delta W_2$$

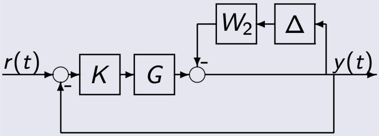
$$1+\tilde{G}K = (1+GK)(1+\Delta W_2\mathcal{T})$$

so wno $\{\,(1+\tilde{G}K)\} = \text{wno}\{(1+GK)\} + \text{wno}\{(1+\Delta W_2\mathcal{T})\}$.
But wno $\{(1+\Delta W_2\mathcal{T})\} = 0$ because $\|\Delta W_2\mathcal{T}\|_\infty < 1$.

Robust stability condition for plants with feedback uncertainty (1) :

$$\tilde{G} = \frac{G}{1 + \Delta W_2} \Rightarrow H = W_2 \frac{-1}{1 + GK}$$

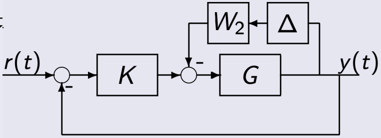
Closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ *iff* $\|W_2 \mathcal{S}\|_\infty < 1$.



Robust stability condition for plants with feedback uncertainty (2) :

$$\tilde{G} = \frac{G}{1 + \Delta W_2 G} \Rightarrow H = W_2 \frac{-G}{1 + GK}$$

Closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ *iff* $\|W_2 G \mathcal{S}\|_\infty < 1$.



Robust Stability

Robust Stability Condition :

The robust stability condition for systems with multiplicative uncertainty is defined as $\boxed{\|W_2\mathcal{T}\|_\infty < 1}$ where $W_2(s)$ is typically a high-pass filter. It guarantees small $|\mathcal{T}(j\omega)|$ at high frequencies, where unmodelled dynamics are large.

Graphical interpretation :

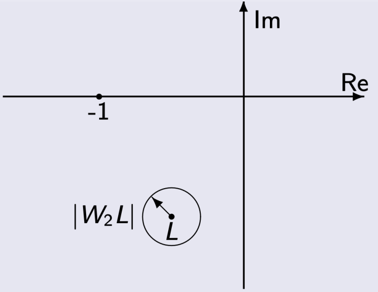
The robust stability condition in the frequency-domain is given by :

$$|W_2(j\omega)\mathcal{T}(j\omega)| < 1 \qquad \forall\omega$$

$$\left|\frac{W_2(j\omega)L(j\omega)}{1+L(j\omega)}\right| < 1 \quad \forall\omega,$$

$$\Leftrightarrow$$

$$|W_2(j\omega)L(j\omega)| < |1+L(j\omega)|, \quad \forall\omega$$



Robust Performance

- Nominal performance condition :

$$\boxed{\|W_1\mathcal{S}\|_\infty < 1}$$

- Robust stability condition for multiplicative uncertainty :

$$\boxed{\|W_2\mathcal{T}\|_\infty < 1}$$

- Robust performance for multiplicative uncertainty :

$$\boxed{\|W_2\mathcal{T}\|_\infty < 1 \quad \text{and} \quad \|W_1\tilde{\mathcal{S}}\|_\infty < 1}$$

where :

$$\tilde{\mathcal{S}} = \frac{1}{1+\tilde{G}K} = \frac{1}{1+GK(1+\Delta W_2)} = \frac{1}{(1+GK)(1+\Delta W_2\mathcal{T})} = \frac{\mathcal{S}}{1+\Delta W_2\mathcal{T}}$$

Robust Performance

Theorem

A necessary and sufficient condition for robust performance of a plant model with multiplicative uncertainty is

$$\| \, |W_1\mathcal{S}| + |W_2\mathcal{T}| \, \|_\infty < 1$$

Proof : (Sufficiency) The above robust performance condition is equivalent to :

$$\|W_2\mathcal{T}\|_\infty < 1 \quad \text{and} \quad \left\| \frac{W_1\mathcal{S}}{1 - |W_2\mathcal{T}|} \right\|_\infty < 1$$

On the other hand : $1 = |1 + \Delta W_2\mathcal{T} - \Delta W_2\mathcal{T}| \leq |1 + \Delta W_2\mathcal{T}| + |W_2\mathcal{T}|$ and therefore $1 - |W_2\mathcal{T}| \leq |1 + \Delta W_2\mathcal{T}|$. This implies that

$$\left\| \frac{W_1\mathcal{S}}{1 - |W_2\mathcal{T}|} \right\|_\infty \geq \left\| \frac{W_1\mathcal{S}}{1 + \Delta W_2\mathcal{T}} \right\|_\infty \quad \Rightarrow \quad \left\| \frac{W_1\mathcal{S}}{1 + \Delta W_2\mathcal{T}} \right\|_\infty < 1$$

Robust Performance

Graphical interpretation :

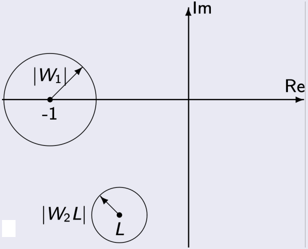
The robust performance condition for systems with multiplicative uncertainty is given by :

$$\| \, |W_1 \mathcal{S}| + |W_2 \mathcal{T}| \, \|_\infty < 1$$

$$\left| \frac{W_1(j\omega)}{1 + L(j\omega)} \right| + \left| \frac{W_2(j\omega)L(j\omega)}{1 + L(j\omega)} \right| < 1 \qquad \forall \omega$$

$$\Leftrightarrow$$

$$|W_1(j\omega)| + |W_2(j\omega)L(j\omega)| < |1 + L(j\omega)|, \quad \forall \omega$$



Limit of Performance

Algebraic Constraints :

- $\mathcal{S} + \mathcal{T} = 1$ therefore :

$$\big||\mathcal{S}(j\omega)| - |\mathcal{T}(j\omega)|\big| \leq \overbrace{|\mathcal{S}(j\omega) + \mathcal{T}(j\omega)|}^{=1} \leq \big||\mathcal{S}(j\omega)| + |\mathcal{T}(j\omega)|\big|$$

So $|\mathcal{S}(j\omega)|$ and $|\mathcal{T}(j\omega)|$ cannot both be less than $1/2$ at the same frequency.

- A necessary condition for robust performance is that :

$$\min\{|W_1(j\omega)|, |W_2(j\omega)|\} < 1, \quad \forall\omega$$

To illustrate this, assume that $|W_1| \leq |W_2|$ at a given frequency. Therefore :

$$|W_1| = |W_1[\mathcal{S} + \mathcal{T}]| \leq |W_1\mathcal{S}| + |W_1\mathcal{T}| \leq |W_1\mathcal{S}| + |W_2\mathcal{T}| \leq 1$$

The same conclusions can be obtained when $|W_2| \leq |W_1|$. So at every frequency either $|W_1|$ or $|W_2|$ must be less than 1. Typically $|W_1|$ is monotonically decreasing and $|W_2|$ is monotonically increasing, thus their intersection should be always below the zero dB axis.

Limit of Performance

Preliminaries

- If p is a pole and z a zero of $L = GK$ both in Re $s \geq 0$ then :

$$\mathcal{S}(p) = 0 \quad \mathcal{S}(z) = 1 \quad \mathcal{T}(p) = 1 \quad \mathcal{T}(z) = 0$$

  because $L(p) = \infty$ and $L(z) = 0$ and therefore : $\mathcal{S}(p) = \dfrac{1}{1 + L(p)} = 0$

- Define $\mathcal{M}$ as the set of stable transfer functions with bounded infinity norm.
  - $F(s) \in \mathcal{M}$ is *all-pass* if $|F(j\omega)| = 1 \quad \forall\omega$
  - $G(s) \in \mathcal{M}$ is *minimum-phase* if it has no zeros in Re $s > 0$.
  - Every transfer function $G \in \mathcal{M}$ can be presented as $G = G_{ap}G_{mp}$

Example

$$G(s) = \frac{(s+1)(s-2)}{(s+3)(s+4)} = \underbrace{\frac{s-2}{s+2}}_{G_{ap}(s)} \underbrace{\frac{(s+2)(s+1)}{(s+3)(s+4)}}_{G_{mp}(s)}$$

Limit of Performance

Theorem (Maximum Modulus Theorem)

Suppose that Ω is a region (nonempty, open, connected set) in the complex plane and F is a function that is analytic in Ω. Suppose that F is not equal to a constant. Then $|F|$ does not attain its maximum value at an interior point of Ω.

A simple application of this theorem, with Ω equal to the open right half-plane, shows that for F in $\mathcal{M}$

$$\|F\|_\infty = \sup_{\mathrm{Re}\ s>0} |F(s)|$$

Limit of Performance

Analytic Constraints :

- Zeros of $L = GK$ in RHP limit the nominal performance :

$$\|W_1\mathcal{S}\|_\infty = \sup_{\text{Re}s\geq 0} |W_1(s)\mathcal{S}(s)| \geq |W_1(z)\mathcal{S}(z)| = |W_1(z)|$$

  - If $|W_1(z)| > 1$, nominal performance cannot be achieved.
  - Since $W_1(s)$ is typically a low-pass filter, a low frequency unstable zero limits the performance more than a high frequency unstable zero.
  - In industry, it is usually said "*the closed-loop bandwidth is limited to the frequency of unstable zeros of the plant model*".

- Unstable poles of $L = GK$ limit the robust stability :

$$\|W_2\mathcal{T}\|_\infty = \sup_{\text{Re}s\geq 0} |W_2(s)\mathcal{T}(s)| \geq |W_2(p)\mathcal{T}(p)| = |W_2(p)|$$

  - if $|W_2(p)| > 1$, robust stability cannot be achieved.
  - Since $W_2(s)$ is typically a high-pass filter, a high-frequency unstable pole limits the robust stability more than a low frequency unstable pole.

Analytic Constraints

Analytic Constraints :

- Zeros and poles of $L = GK$ in RHP limit the **nominal performance** significantly. We have :

$$\mathcal{S} = \frac{1}{1+\bar{L}\frac{s-z}{s-p}} = \frac{s-p}{(s-p)+\bar{L}(s-z)} = \overbrace{\frac{s-p}{s+p}}^{\mathcal{S}_{ap}} \overbrace{\frac{s+p}{(s-p)+\bar{L}(s-z)}}^{\mathcal{S}_{mp}}$$

On the other hand $\mathcal{S}(z) = \mathcal{S}_{ap}(z)\mathcal{S}_{mp}(z) = 1 \Rightarrow \mathcal{S}_{mp}(z) = \mathcal{S}_{ap}^{-1}(z)$, Then :

$$\|W_1\mathcal{S}\|_\infty = \|W_1\mathcal{S}_{mp}\|_\infty \geq |W_1(z)\mathcal{S}_{mp}(z)| = \left|W_1(z)\frac{z+p}{z-p}\right|$$

  - Unstable pole and zero close to each other limits significantly the achievable performance.
  - The worst situation is when they are both in low frequencies (because $W_1(s)$ is a low pass filter).

Analytic Constraints

Analytic Constraints :

- Zeros and poles of $L = GK$ in RHP limit the **robust stability** significantly. We have :

$$\mathcal{T} = \frac{\bar{L}\frac{s-z}{s-p}}{1+\bar{L}\frac{s-z}{s-p}} = \frac{\bar{L}(s-z)}{(s-p)+\bar{L}(s-z)} = \overbrace{\frac{s-z}{s+z}}^{\mathcal{T}_{ap}}\overbrace{\frac{s+z}{(s-p)+\bar{L}(s-z)}}^{\mathcal{T}_{mp}}$$

On the other hand $\mathcal{T}(p) = \mathcal{T}_{ap}(p)\mathcal{T}_{mp}(p) = 1 \Rightarrow \mathcal{T}_{mp}(p) = \mathcal{T}_{ap}^{-1}(p)$, Then :

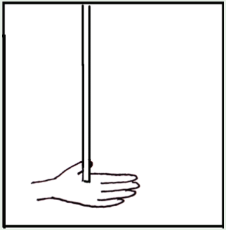
$$\|W_2\mathcal{T}\|_\infty = \|W_2\mathcal{T}_{mp}\|_\infty \geq |W_2(p)\mathcal{T}_{mp}(p)| = \left|W_2(p)\frac{p+z}{p-z}\right|$$

  - Unstable pole and zero close to each other limits significantly the robust stability.
  - The worst situation is when they are both in high frequencies (because $W_2(s)$ is a high-pass filter).

Analytic Constraints

Example (Balancing a stick by hand)

If we want to balance a stick on our hand :



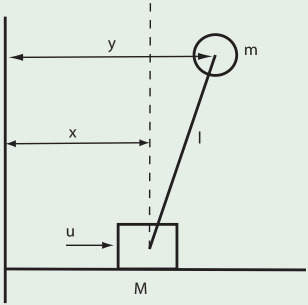
- We choose
  (A) A long stick **(B)** A short stick
- We add a mass on the top of the stick
  (A) No way **(B)** Off course
- We look at
  (A) Our hand **(B)** the top of the stick

Analytic Constraints

Example

Consider the inverse pendulum problem.

$$
\begin{aligned}
(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) &= u \\
m(\ddot{x}\cos\theta + l\ddot{\theta} - g\sin\theta) &= d
\end{aligned}
$$



Linearized model :

$$
\begin{pmatrix} x \\ \theta \end{pmatrix} = \frac{1}{s^2[Mls^2 - (M+m)g]} \begin{pmatrix} ls^2 - g & -ls^2 \\ -s^2 & \frac{M+m}{m}s^2 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}
$$

Analytic Constraints

Example

Measuring x **:** (looking at the hand)

$$T_{ux} = \frac{ls^2 - g}{s^2[Mls^2 - (M+m)g]}$$

$$\text{RHP poles and zeros : } z = \sqrt{g/l} \quad p = 0, 0, \sqrt{\frac{(M+m)g}{Ml}}$$

From a robust stability perspective :

- It is a very difficult problem when m/M is small (unstable pole and zero are too close).
- By increasing m/M (for a fixed l) the situation improves ($\frac{p+z}{p-z}$ is decreased) but there is a trade-off because $W_2(p)$ will increase as well.
- The best scenario is to increase l and m/M.

Analytic Constraints

Example

Measuring y **:** (looking at the top)

$$T_{uy} = \frac{-g}{s^2[Mls^2 - (M+m)g]}$$

From a robust stability perspective :

- Since there is no RHP zero the system is much easier to stabilize.
- A larger l gives a smaller p so the system can be better controlled.
- The choice of the place of sensors changes the zeros of the system can affects significantly the limit of performance.

Analytic Constraints

Theorem (The Waterbed Effect)

Suppose that G has a zero at z with $Re\, z > 0$ and :

$$M_1 := \max_{\omega_1 \leq \omega \leq \omega_2} |\mathcal{S}(j\omega)| \qquad M_2 := \|\mathcal{S}\|_\infty$$

Then there exist positive constants c_1 and c_2, depending only on ω_1, ω_2 and z, such that :

$$c_1 \log M_1 + c_2 \log M_2 \geq \log |\mathcal{S}_{ap}^{-1}(z)| \geq 0$$

- Note that $|\mathcal{S}_{ap}^{-1}(z)| = 1$ if L has no unstable pole. In this case $\log |\mathcal{S}_{ap}^{-1}(z)| = 0$.
- If L has one unstable pole then $\log |\mathcal{S}_{ap}^{-1}(z)| \geq 0$.
- To obtain better performance M_1 should be reduced. The theorem shows this necessarily leads to an increase of $M_2 = \|\mathcal{S}\|_\infty$ (waterbed effect) and reduces the modulus margin.

Analytic Constraints

Theorem (The Area Formula)

Assume that the relative degree of L is at least 2. Then if the closed-loop system is stable :

$$\int_0^\infty \log|\mathcal{S}(j\omega)|d\omega = \pi(\log e)\sum_i Re\, p_i$$

where $\{p_i\}$ *denotes the set of poles of L in* $Re\ s > 0$.

- For a stable L, the right hand side is equal to zero. So the area of disturbance attenuation is equal to the area of disturbance amplification.
- For unstable L, it is more difficult to improve the performance.
- In contrast with the waterbed effect (which concerns only non-minimum phase systems), improving the performance in some frequency (decreasing M_1) will not necessarily increase $M_2 = \|\mathcal{S}\|_\infty$.