Numerical Methods in Biomechanics

ME-484

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EPFL - Laboratory of Biomechanical Orthopedics

Content - schedule

W01: Organization, introduction and examples
W02: External lecturers
W03: Partial Differential Equations
W04: Solid mechanics in numerical biomechanics
W05: Fluid mechanics in numerical biomechanics
W06: The Finite Element Method (FEM), and its extensions
W07: Midterm project presentations
W08: Midterm evaluation
W09: Multiphysics and coupling
W10: Example 1
W11: Example 2
W12: Example 3
W13: Final project presentation

Solid mechanics in biomechanics

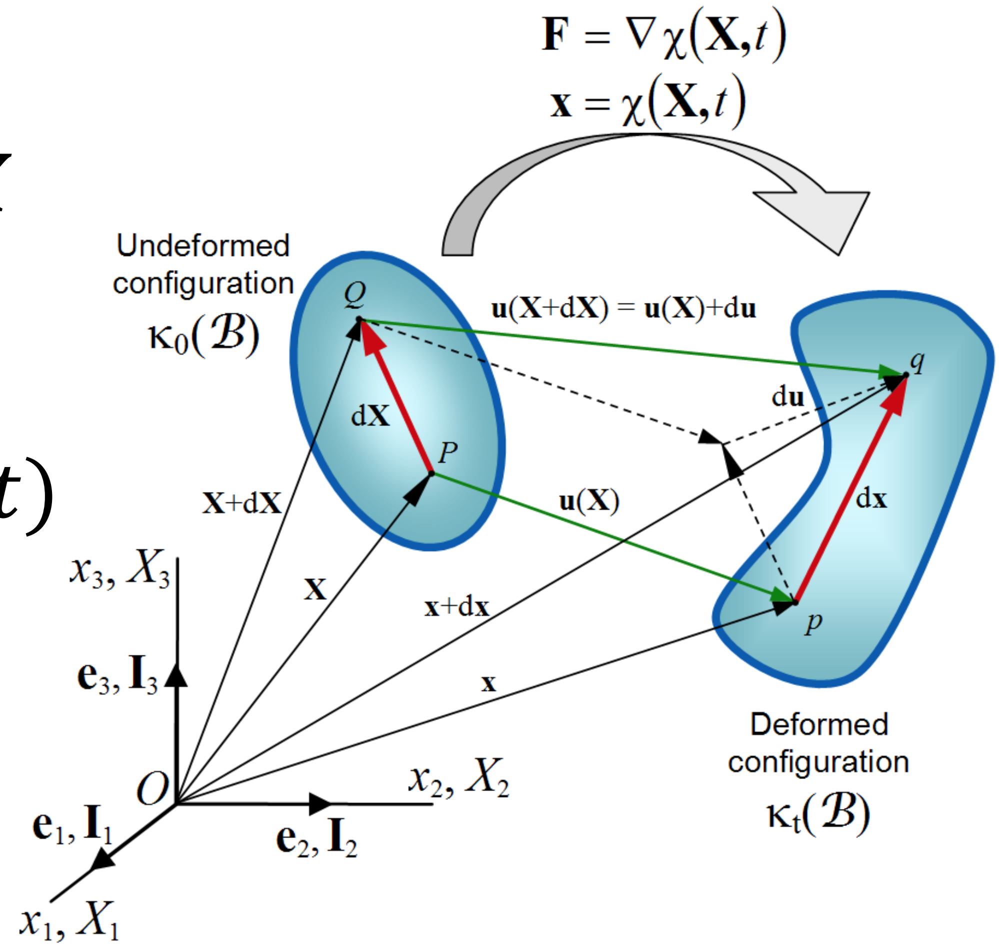
- Large strain (soft tissues)
- Non-linear continuum mechanics
- Strain contains a nonlinear term
- Several stress definitions (different)
- Stress-strain relationship (constitutive law) build from thermodynamically admissible potentials

Strain

Lagrangian description

Displacement $\boldsymbol{u}$ of point $\boldsymbol{X}$ to a position $\boldsymbol{x}$

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t)$$



https://en.wikipedia.org/wiki/Finite_strain_theory

Strain

- Deformation gradient tensor $\mathbf{F}$

$$\mathbf{F} = d\boldsymbol{x}/d\boldsymbol{X} = grad(\boldsymbol{u}) + \mathbf{1} = \nabla_X \boldsymbol{u} + \mathbf{1}$$

$$J = \det(\mathbf{F}), dv = JdV$$ (Jacobian determinant)

- Right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$
- Green-Lagrange strain tensor $\mathbf{E} = 1/2(\mathbf{C} - \mathbf{1})$

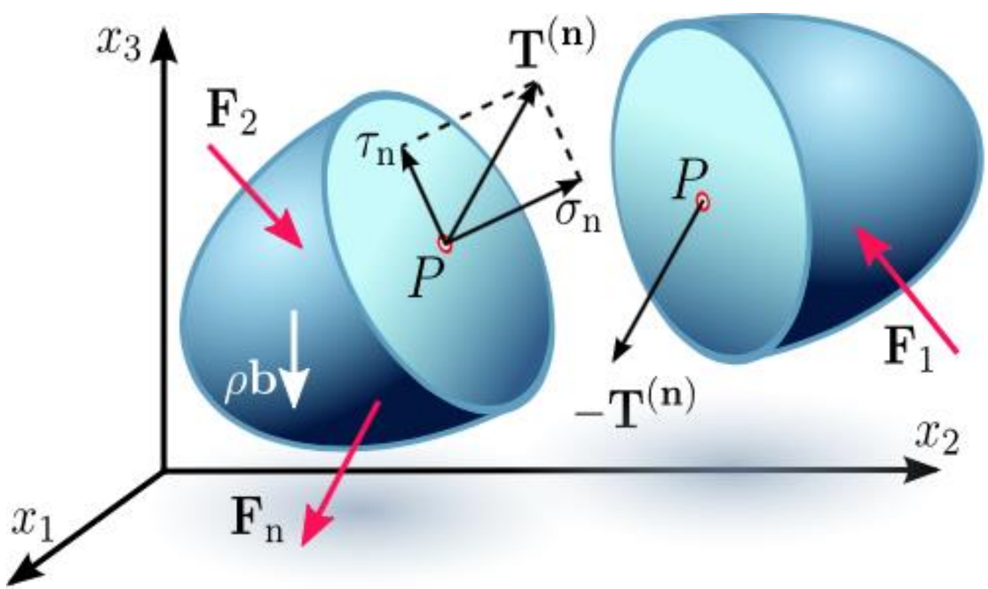
$$\mathbf{E} = \frac{1}{2}[(\nabla_{\boldsymbol{X}}\boldsymbol{u})^T + \nabla_{\boldsymbol{X}}\boldsymbol{u} + (\nabla_{\boldsymbol{X}}\boldsymbol{u})^T\nabla_{\boldsymbol{X}}\boldsymbol{u}]$$

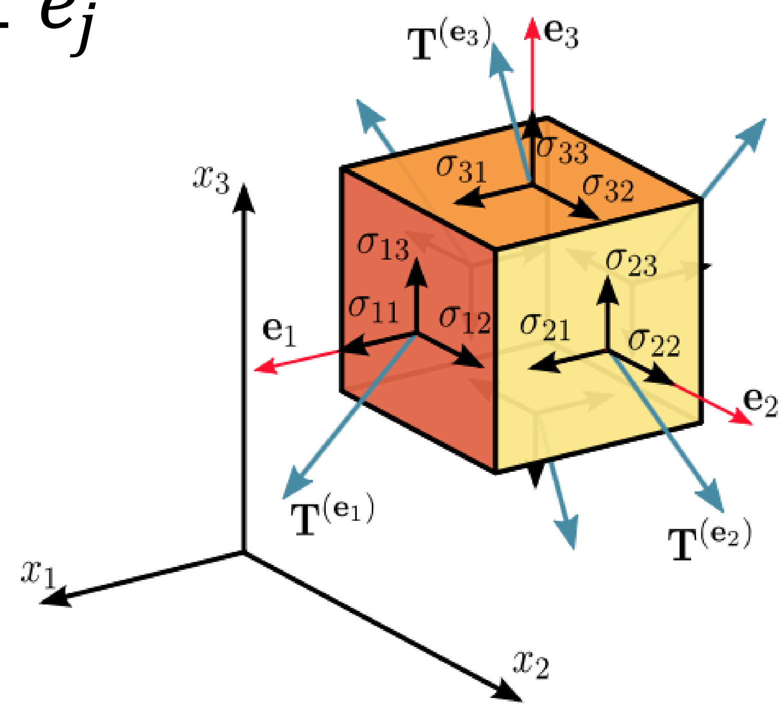
Stress

<u>Cauchy's stress theorem</u>

Stress state at any point P represented by (**T**,***n***) or **σ**

$$\sigma_{ij} = \frac{\text{Force } T_i}{\text{surface} \perp e_j}$$





https://en.wikipedia.org/wiki/Cauchy_stress_tensor

Stress

- Spatial or true (Cauchy) stress tensor $\mathbf{s} = f/a$ $(= \boldsymbol{\sigma})$
  current force with current area
- Nominal (1st Piola-Kirchhoff) stress tensor $\mathbf{P} = f/a_r$
  current force with reference area
- Material (2nd Piola-Kirchhoff) stress tensor $\mathbf{S} = f_r/a_r$
  reference force with reference area

$$\mathbf{S} = \mathbf{F}^{-1}\,\mathbf{P}$$

$$\mathbf{s} = J^{-1}\,\mathbf{P}\,\mathbf{F}^{T} = J^{-1}\,\mathbf{F}\,\mathbf{S}\,\mathbf{F}^{T}$$

Small strain

- Small strain $\boldsymbol{\varepsilon} = \frac{1}{2}[(\nabla_{\boldsymbol{X}}\boldsymbol{u})^T + \nabla_{\boldsymbol{X}}\boldsymbol{u} + \cancel{(\nabla_{\boldsymbol{X}}\boldsymbol{u})^T\nabla_{\boldsymbol{X}}\boldsymbol{u}}]$
- All strain/stress definitions are equal
- Often associated with linear elasticity

Linear elasticity

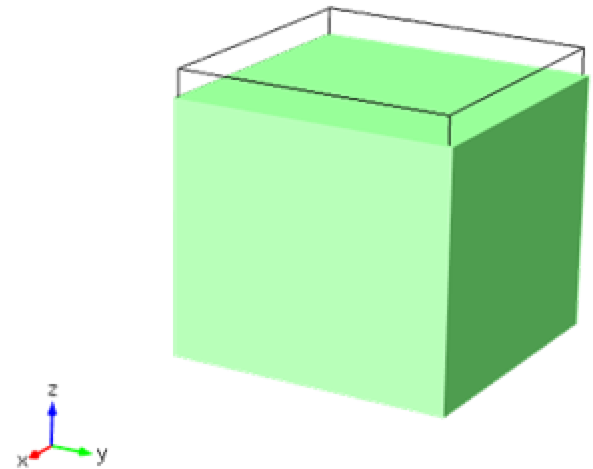
- Linear constitutive law (Hook's law)

$$\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon}$$

- Elastic tensor (4 order) $\mathbf{D} \Longrightarrow$ 21 parameters
- Isotropy $\Longrightarrow$ only 2 parameters required
  – Young modulus E, Poisson ratio ν
  – Lamé parameters λ, μ
  – Bulk modulus K, shear modulus G

https://en.wikipedia.org/wiki/Lamé_parameters

Example: bone sample



- Cube 1 cm
- E = 1 GPa (Young's modulus)
- $\nu = 0.3$ (Poisson's ratio)
- $u_z = -10\ \mu m$ (imposed displacement)
- $F_{zz} = 0.999$ (deformation gradient)
- $\varepsilon_{zz} = -0.001 = -0.1\% = -1'000\ \mu\varepsilon$ (axial strain)
- $\sigma_{zz} = E\ \varepsilon_{zz} = -1$ MPa (axial stress)
- Force = -100 N (applied force)

Stress & strain invariants

- Strain and stress are symmetric tensor (6 scalars)
  Balance of angular momentum (Newton's 2nd law)
- Tensor of rank 2 $\Rightarrow$ 3 (independent) invariants
- Invariants of tensor provide a scalar measure
- Invariants are used in constitutive laws
- Invariants are related to failure
  - Von Mises stress for ductile (bone, metal)
  - Maximum principal stress for brittle material (cement)

Strain invariants

- $\mathrm{I}_1 = \mathrm{tr}(\mathbf{C})$ (volumetric strain)
- $\mathrm{I}_2 = \frac{1}{2}(\mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2))$ (deviatoric strain)
- $\mathrm{I}_3 = \det(\mathbf{C}) = \det(\mathbf{F}^{\mathrm{T}}\mathbf{F}) = \det(\mathbf{F}^2)$

$$= \left(\frac{dv}{dV}\right)^2 = \left(\frac{d\rho_0}{d\rho}\right)^2$$ (volume change)

Mass conservation

I_3 = 1 if the material is incompressible

Stress invariants

- $J_1 = \mathrm{tr}(\mathbf{s})$ (hydrostatic pressure: $-⅓J_1$)
- $J_2 = ½(\mathrm{tr}(\mathbf{s}^2) - \mathrm{tr}(\mathbf{s})^2)$ (von Mises: $\sqrt{3J_2}$)
- $J_3 = \det(\mathbf{s})$

Principal strain (invariants)

- Coordinate system where **C** becomes diagonal
- Eigenvalues of strain tensor **C**: C_1, C_2, C_3
- Maximum principal strain: max(C_i) (tension)
- Minimum principal strain: min(C_i) (compression)

Principal stress (invariants)

- Coordinate system where **s** becomes diagonal
- Eigenvalues of stress tensor **s**: λ_1 , λ_2, λ_3
- Maximum principal stress: max (λ_i) (tension)
- Minimum principal stress: min(λ_i) (compression)

Equilibrium equation (PDE)

- Equilibrium equation $\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{f}_V - \nabla_{\boldsymbol{x}} \cdot \mathbf{s}$
  Balance of linear momentum
- Displacement $\boldsymbol{u}$ is the dependent variable
- Constitutive law: $\mathbf{s} = \mathbf{s}(\boldsymbol{u})$ $\Leftrightarrow$ $\mathbf{S} = \mathbf{D}:\mathbf{E}(\boldsymbol{u})$
- Boundary conditions:
  displacement, force, stress, strain
- Initial conditions:
  reference (material) configuration

Constitutive law

- Constitutive law relates stress ($\boldsymbol{S}$) to strain ($\boldsymbol{C}$)
- Strain energy potential W function of $\boldsymbol{C}$ invariants (I_1, I_2, I_3)

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}} = \frac{\partial W}{\partial \mathbf{E}}$$

- Neo-Hook: $W = \frac{1}{2}\mu(I_1 - 3) - \underbrace{\mu\log(J) + \frac{1}{2}\lambda[\log(J)]^2}_{0 \text{ if } J_{el} = J = \det(\boldsymbol{F}) = 1}$
- Constitutive law thermodynamically admissible

Constitutive law

- Nonlinear Elasticity (hyperelasticity)
- Plasticity: permanent deformation
- Viscosity: time dependent effects
- Damage: fracture, failure, cyclic loading
- Non-homogeneity: space dependent properties
- Non-isotropy: direction dependent properties

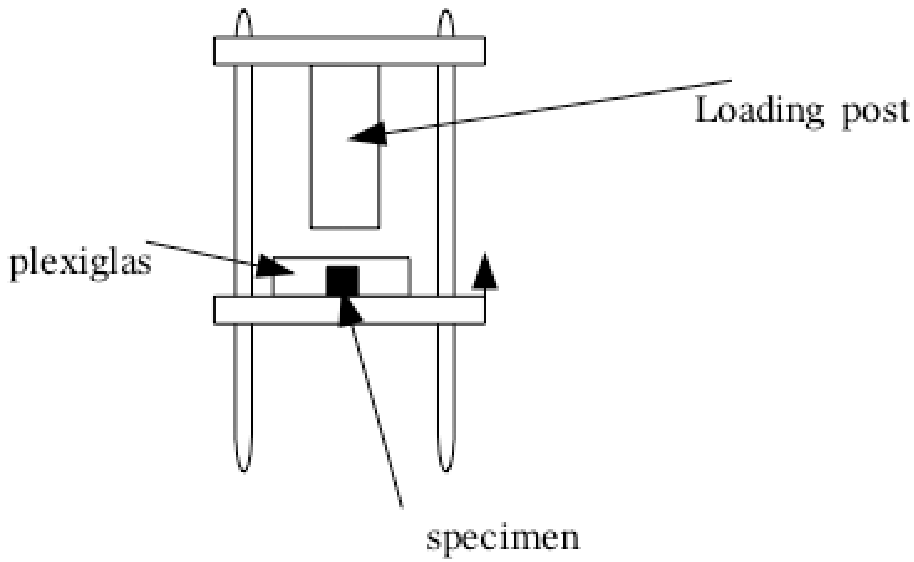
2D approximation

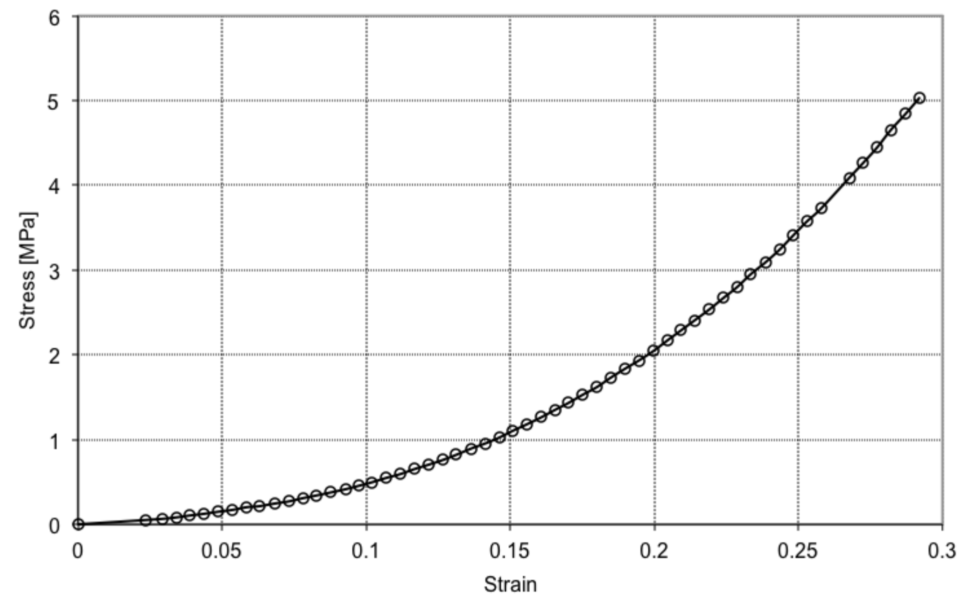
- Cylindrical symmetry
  $\Rightarrow$ no angular dependency

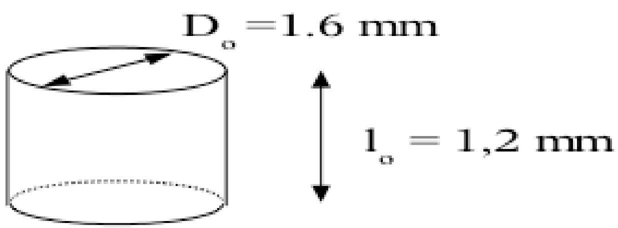
- Plane stress: out of plane stress is negligible
  $\Rightarrow$ thin plate, with in-plate loading

- Plane strain: out of plane strain is negligible
  $\Rightarrow$ thick plate, (or axial displacement restricted)

Example: cartilage sample







Strain = 0

Strain = 20%

Humeral head cartilage sample

Example: cartilage sample

- 2D axisymmetric
- Hook (small strain) vs. hyperelastic (Neo-Hook)
- Hook: $E = 5\ \mathrm{MPa}, \nu = 0.4$
- Neo-hook: $W = \frac{1}{2}\mu(I_1 - 3) - \mu\log(J) + \frac{1}{2}\lambda[\log(J)]^2$

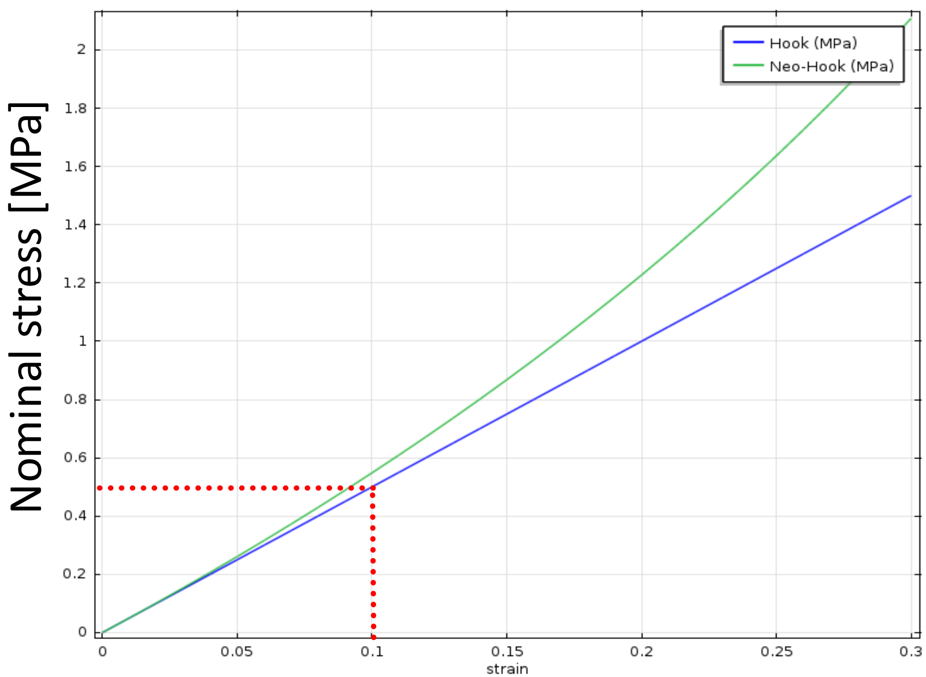
$$\mu = \frac{E}{2(1+\nu)} = 1.79\ [\mathrm{MPa}]$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 7.14\ [\mathrm{MPa}]$$

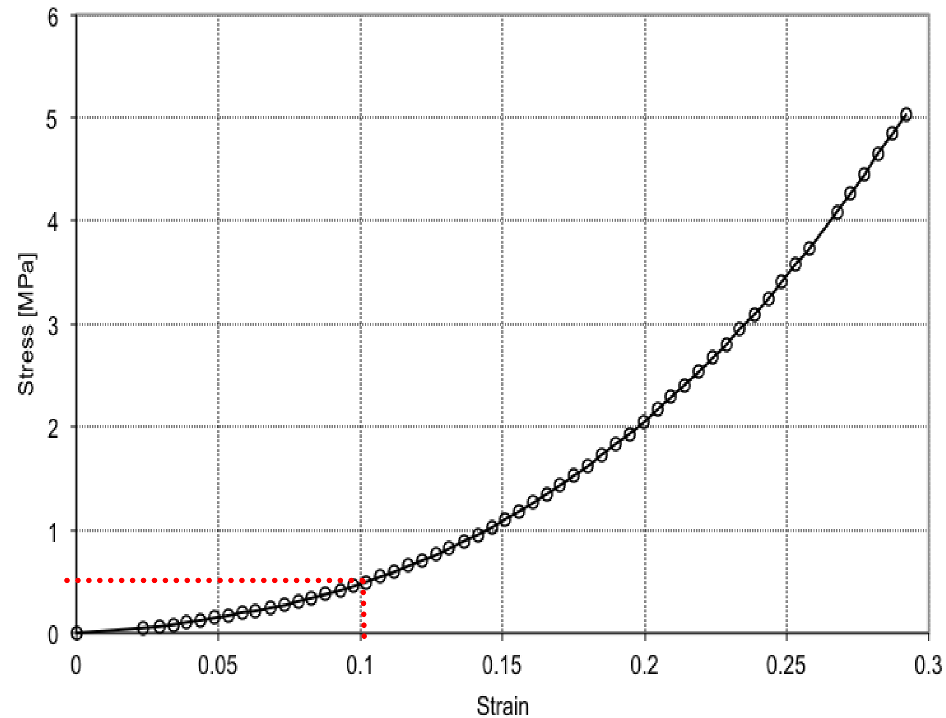
Solid mechanics

Stress = Young’s modulus · Strain = 5 MPa · 10% = 0.5 MPa

Force = Stress · Surface = 0.5 · Pi · $0.8^2 \cong 1$ N

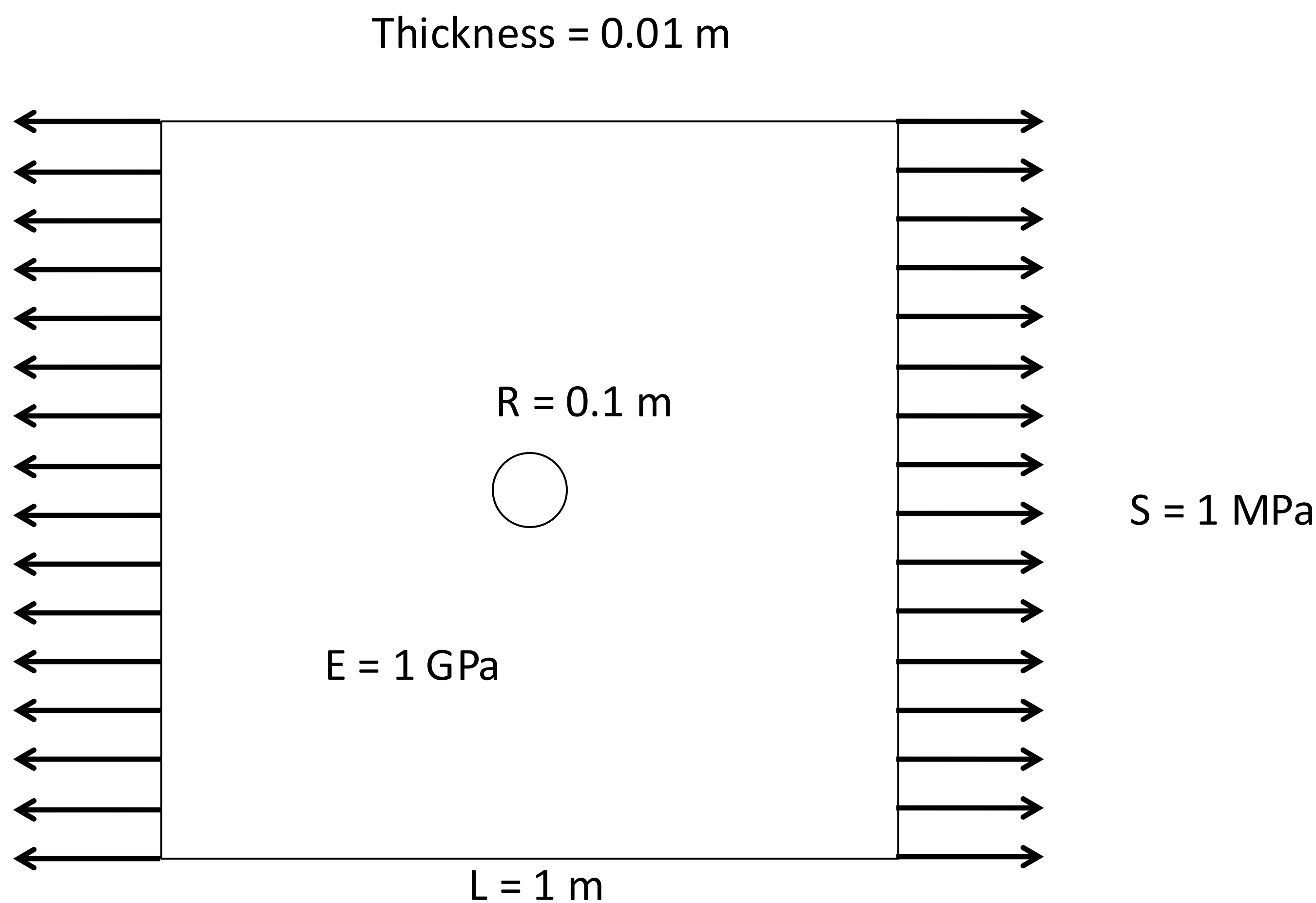


Model predictions



Experimental measurements

3D vs. plane stress



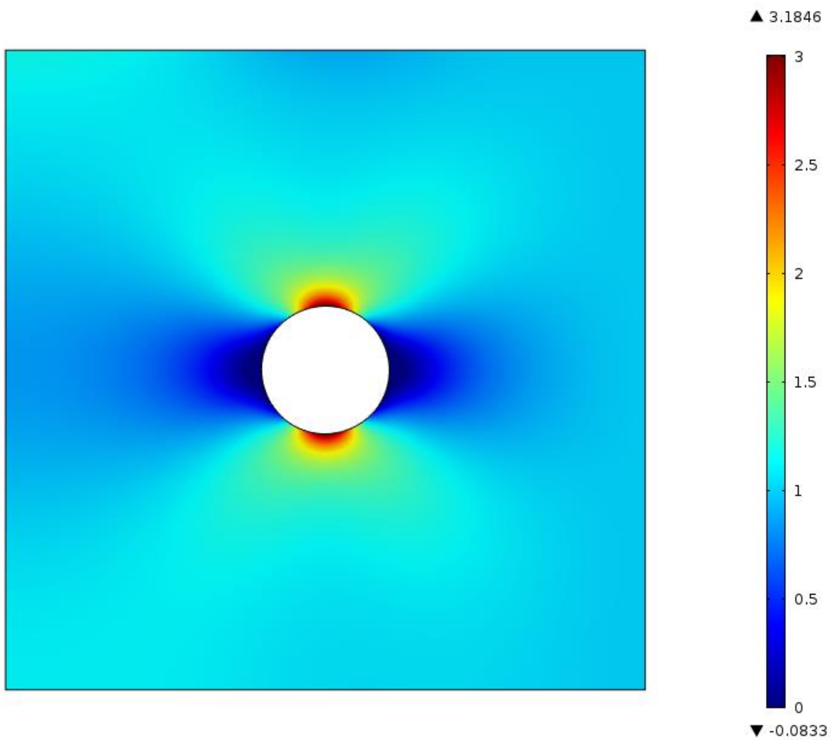
Kirsch plate problem

Kirsch, 1898, Die Theorie der Elastizität und die Bedürfnisse der Festigkeitslehre. Zeitschrift des Vereines deutscher Ingenieure, 42, 797–807

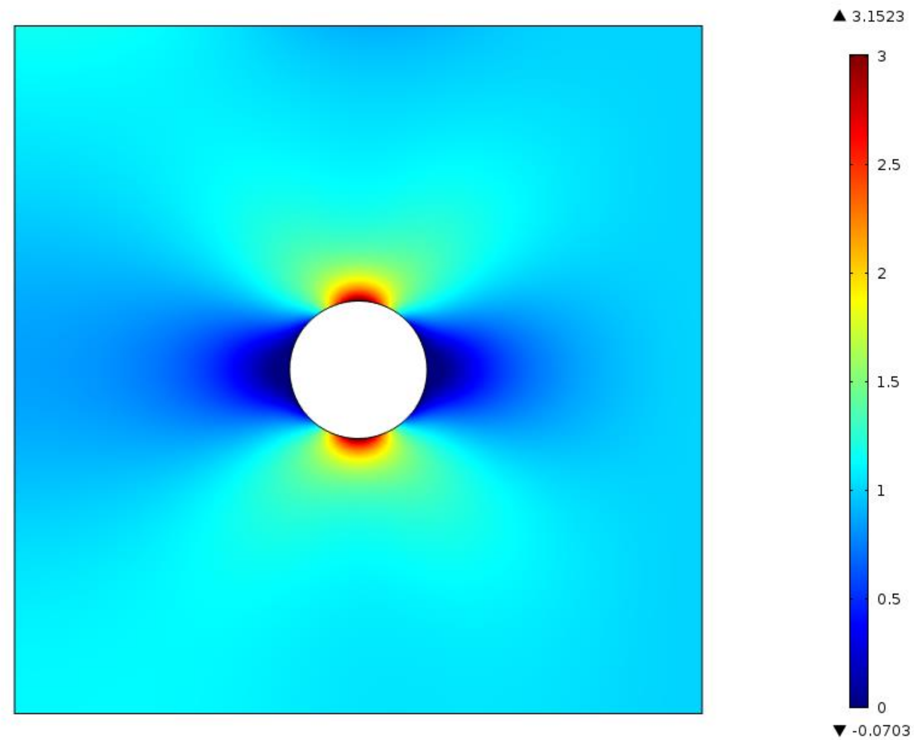
3D vs. plane stress

S_{xx}

$\xrightarrow{x}$



3D



2D

References

- Mécanique des solides déformables, Curnier
- Computational Methods in Solid Mechanics, Curnier
- Méthodes numériques en mécanique des solides, Curnier
- Mécanique des milieux continus: une introduction, Botsis & Deville
- Introduction à la mécanique des solides et structures, Del Pedro, Gmür, Botsis
- Applied Solid Mechanics, Howell, 2008
- Nonlinear Solid Mechanics: a Continuum Approach for Engineering, Holzapfel
- The Feynman Lectures on Physics, 1964 (www.feynmanlectures.info)