Numerical Methods in Biomechanics

ME-484

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EPFL - Laboratory of Biomechanical Orthopedics

Content - schedule

W01: Organization, introduction and examples
W02: External lecturers
W03: Partial Differential Equations
W04: Solid mechanics in numerical biomechanics
W05: Fluid mechanics in numerical biomechanics
W06: Midterm project presentations
W07: The Finite Element Method (FEM), and its extensions
W08: Midterm evaluation
W09: Multiphysics, coupling, Comsol
W10: Multiphysics example 1
W11: Multiphysics example 2
W12: Multiphysics example 3
W13: Final project presentation

Fluid mechanics

- Eulerian description (reference frame)
  - Fluid properties (velocity, pressure, …) considered at “fixed” points in space
  - Quantities not “followed”, as in Lagrangian description (solid)
- No moving boundaries (of the domain)

Navier-Stokes

Momentum conservation $$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I} + \tau] + \mathbf{F}$$

Mass conservation $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

State equation $$p = p(\rho)$$

Constitutive equation $$\tau = \mu\dot{\gamma} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\mathbf{I}$$

5 dependent variables ($\mathbf{u}$, p, ρ) for 5 PDE

Navier-Stokes

inertial acceleration

Momentum conservation $$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I}+\tau]+\mathbf{F}$$

0 if stationary flow

Navier-Stokes

convection

Momentum conservation $$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I}+\tau]+\mathbf{F}$$

Navier-Stokes is nonlinear

Navier-Stokes

pressure gradient

Momentum conservation $$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I} + \tau] + \mathbf{F}$$

Navier-Stokes

volume forces

Momentum conservation $$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I}+\tau]+\mathbf{F}$$

Navier-Stokes

viscous stress tensor

Momentum conservation $$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I}+\tau]+\mathbf{F}$$

Constitutive equation $$\tau = \mu\dot{\gamma}-\frac{2}{3}\mu(\nabla\cdot\mathbf{u})\mathbf{I}$$

Shear strain rate $$\dot{\gamma} = (\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$$

Shear rate $$\dot{\gamma} = |\dot{\gamma}| = \sqrt{\frac{1}{2}\dot{\gamma}:\dot{\gamma}}$$

Dynamic viscosity $$\mu$$

Kinematic viscosity $$\frac{\mu}{\rho}$$

Navier-Stokes

Momentum conservation $$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I} + \tau] + \mathbf{F}$$

Constitutive equation $$\tau = \mu\dot{\gamma} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\mathbf{I}$$

Shear strain rate $$\dot{\gamma} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

Conservation of mass $$\underbrace{\frac{\partial\rho}{\partial t}}_{\text{0 if incompressible } (\rho=\text{cte})} + \nabla\cdot(\rho\mathbf{u}) = 0$$

0 if **incompressible** (ρ=cte) $\Longrightarrow$ $\nabla\cdot\mathbf{u} = 0$

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I} + \underbrace{\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)]}_{\mu\nabla^2\mathbf{u}} + \mathbf{F}$$

Navier-Stokes

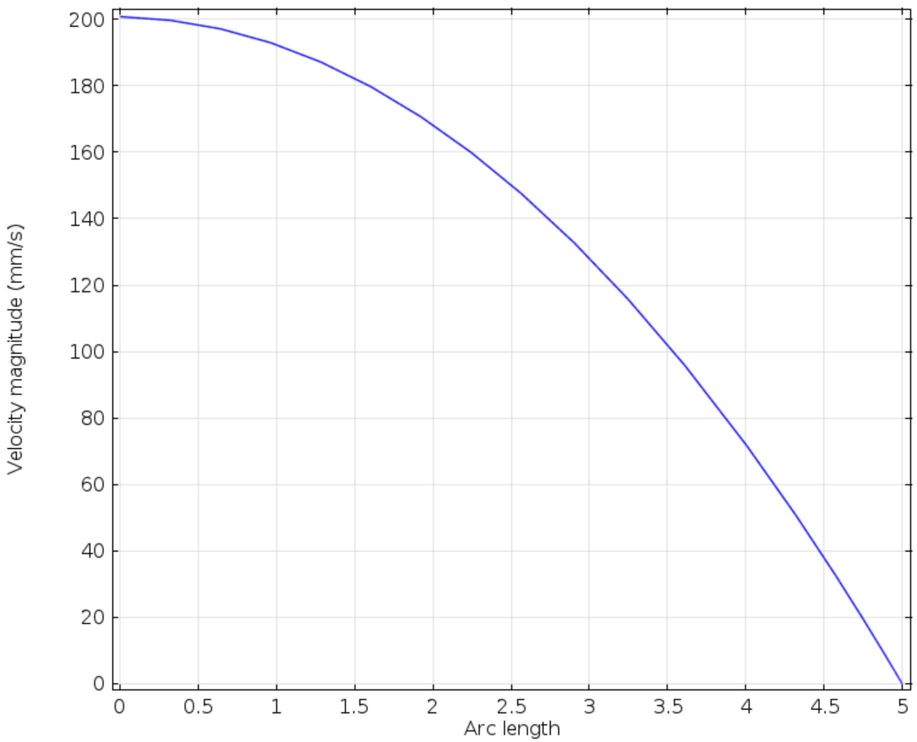
- Typical boundary conditions:
  - Walls no slip: $\mathbf{u} = 0$
  - Inlet: $\mathbf{u}$ or p fixed
  - Outlet: $\mathbf{u}$ or p fixed
- Typical initial conditions
  - $\mathbf{u}(\mathrm{t_o}) = \mathbf{u}_0 = 0$
  - $p(\mathrm{t_o}) = p_0 = 1$ [atm]
  - $\rho(\mathrm{t_o}) = \rho =$ constant

Simplified artery (Poiseuille) flow

- Cylinder (2D axisymmetric)
- Diameter: 10 mm
- Length: 50 mm
- $\rho = 10^3$ kg/m^3 and $\mu = 3 \cdot 10^{-3}$ Pa·s
- Inlet velocity: 100 mm/s (average laminar)
- Outlet (relative) pressure: 100 mmHg
- Stationary (laminar) flow
- Profiles of $\mathbf{u}$ and p

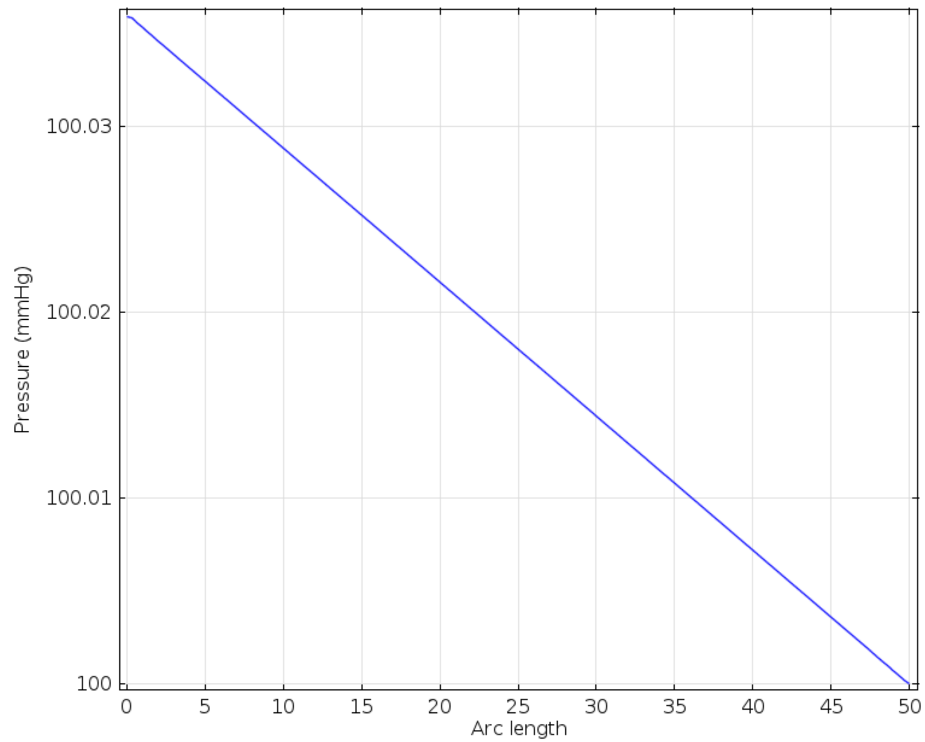
Simplified artery flow

Parabolic velocity profile
(fully developed)



Radial axis

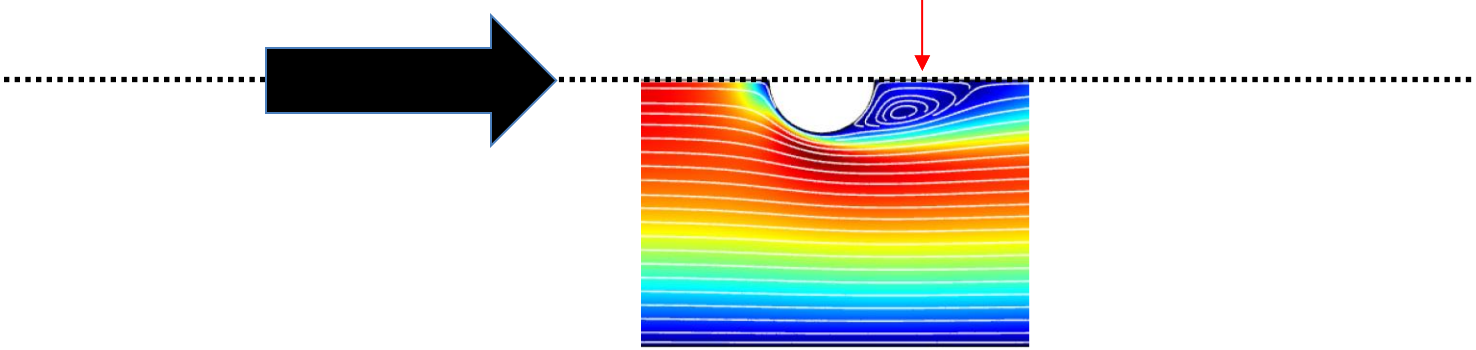
Linear pressure drop



Longitudinal axis

Simplified artery flow with obstacle

- Add spherical obstacle
  - Radius: 1 mm
  - Distance from inlet: 10 mm
- Superpose streamlines on **u**
- Plot profiles of **u** and *p*
- Change inlet velocity



Shear stress

- Boundary with no slip condition
- Tangential stress applied by the fluid on the wall
- Proportional to viscosity and shear rate $\tau_{\mathrm{wall}} = \mu \, \dot{\gamma}_{wall}$
- Arterial wall shear stress (~ 1 Pa)

Mach number

$$\mathbf{Ma} = \frac{|\mathbf{u}|}{a}$$

(a: speed of sound)

- Characterize compressibility
- $\mathrm{Ma} = 0$ for impressible fluid
- Dimensionless number

Reynolds number

$$\mathrm{Re} = \frac{\rho U L}{\mu}$$

(U: typical velocity)
(L: typical length)

- Represented the ratio inertia/viscosity forces
- Characterize turbulence (caused by inertia)
- Dimensionless number
- Local cell Reynolds: $\mathrm{Re}^{\mathrm{c}} = \frac{\rho |\boldsymbol{u}| h}{2\mu}$, ($h$: element size)

With last example

- Evaluate flow rate $Q = \iint_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{ds}$
- Evaluate wall shear stress τ_{wall}
- Evaluate cell Reynolds number *Rec*
- Evaluate "cylinder" Reynolds number Re = $V{\cdot}D/\nu$

  V : average velocity (V = Q/S, S : tube section)

  D : cylinder diameter

  ν : kinematic viscosity

Flow types

- Incompressible (ρ = constant, $\nabla \cdot \mathbf{u} = 0$, Ma = 0)
- Laminar (streamline): layers sliding over one another
- Turbulent: (Re $\gtrsim$ 4000)
- Non-Newton (fluid): viscosity depends on shear rate
- Euler (inviscid): no viscosity
- Stokes flow: no inertia (N-S becomes linear), Re << 1
- Irrotational: no vorticity ($\nabla \times \mathbf{u} = 0$)
- Non-isothermal: temperature dependent variable
- Poiseuille: $\Delta P = \frac{8\mu L Q}{\pi r^4}$

Non-newtonian blood flow

- Viscosity decreases with shear rate

  $0.1 < \dot{\gamma} < 1000$ 1/s

- Viscosity increases with vessel diameter (< 1 mm)

  Fahraeus-Lindqvist effect

Viscosity increases with hematocrit

  Ht : 40→70% ⇒ μ approximately doubles

- Viscosity decreases with temperature (∼2%/℃)

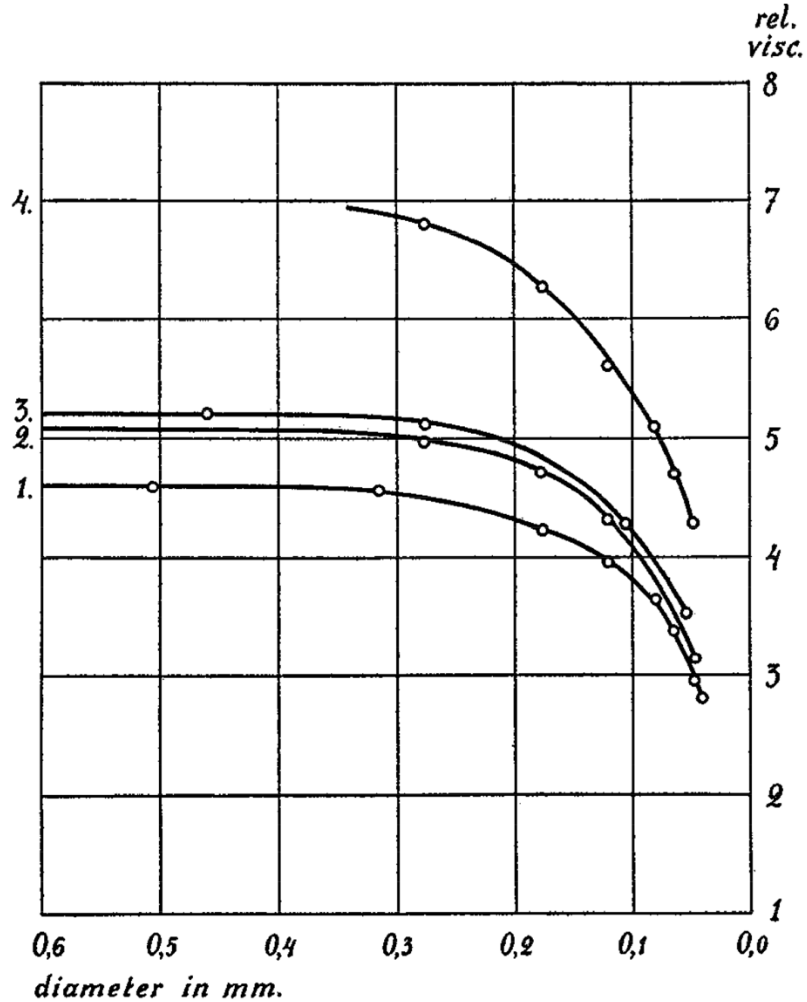
  cold foot / brain

THE VISCOSITY OF THE BLOOD IN NARROW CAPILLARY TUBES

ROBIN FÅHRÆUS AND TORSTEN LINDQVIST

From the Pathological Institute, Uppsala, Sweden

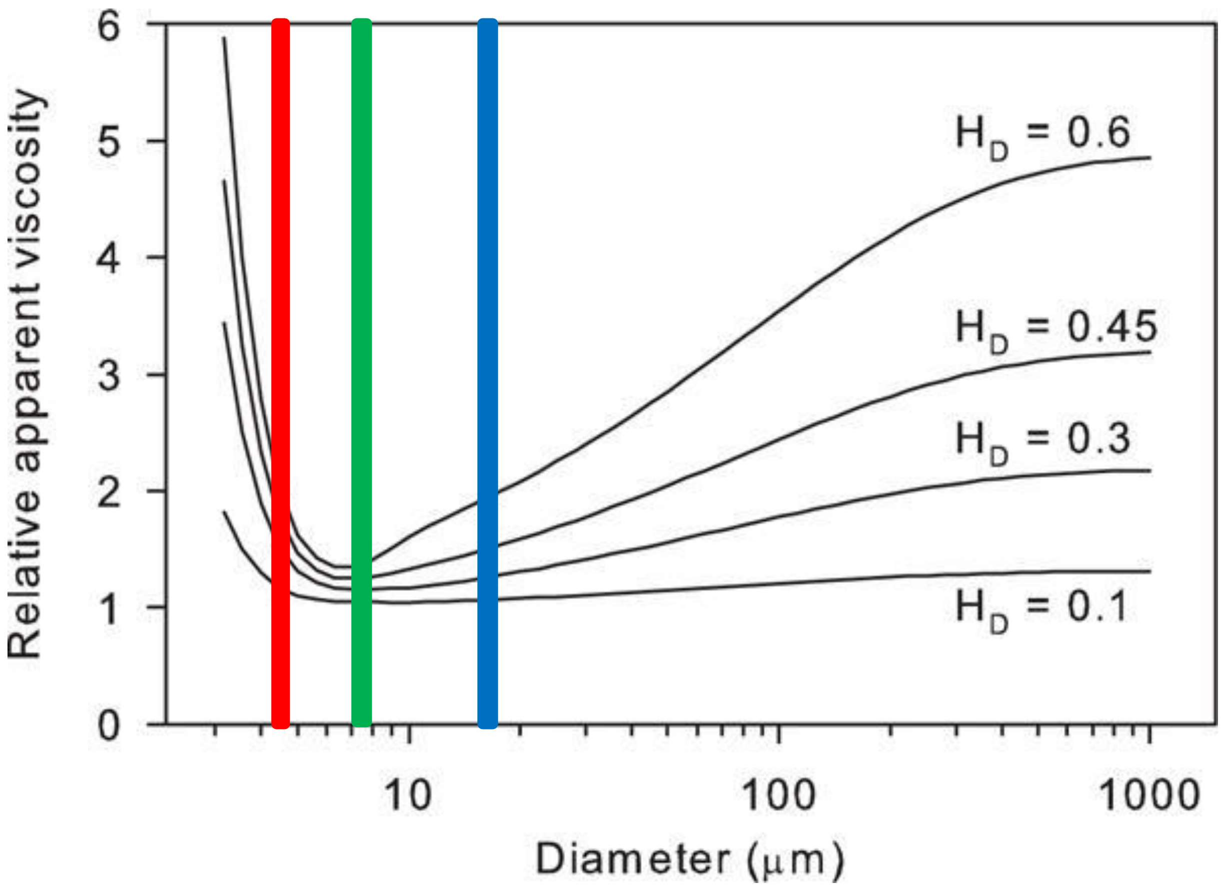
Received for publication December 6, 1930

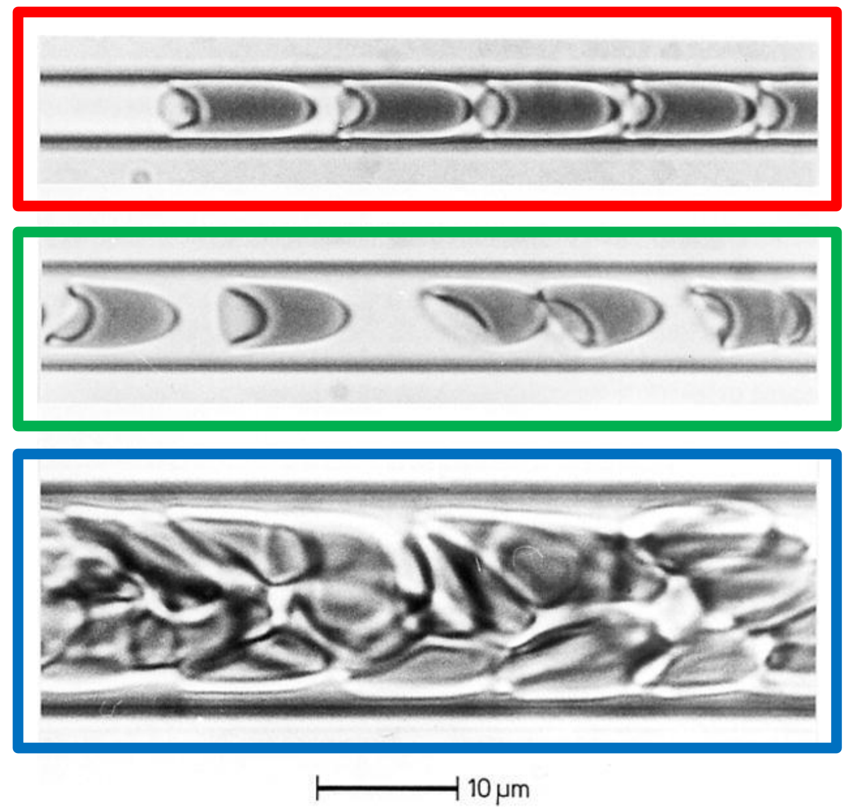


Relative viscosity for 4 series with different reference plasma viscosity

https://journals.physiology.org/doi/pdf/10.1152/ajplegacy.1931.96.3.562

Viscosity vs diameter & hematocrit

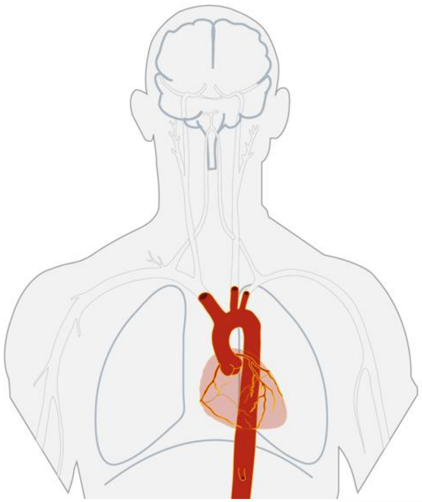




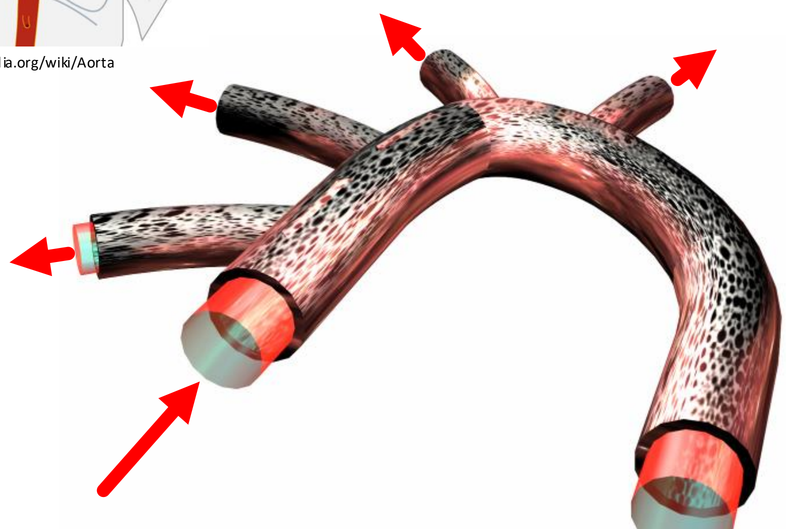
https://doi.org/10.1016%2Fj.crhy.2013.04.002

Hydrodynamics of Capsules and Cells (2003)
Secomb, T. W., and C. Pozrikidis

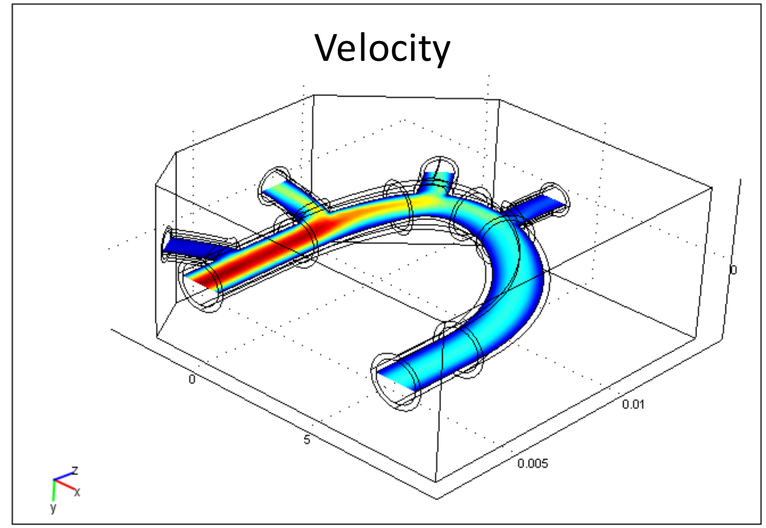
Dynamic flow in aorta

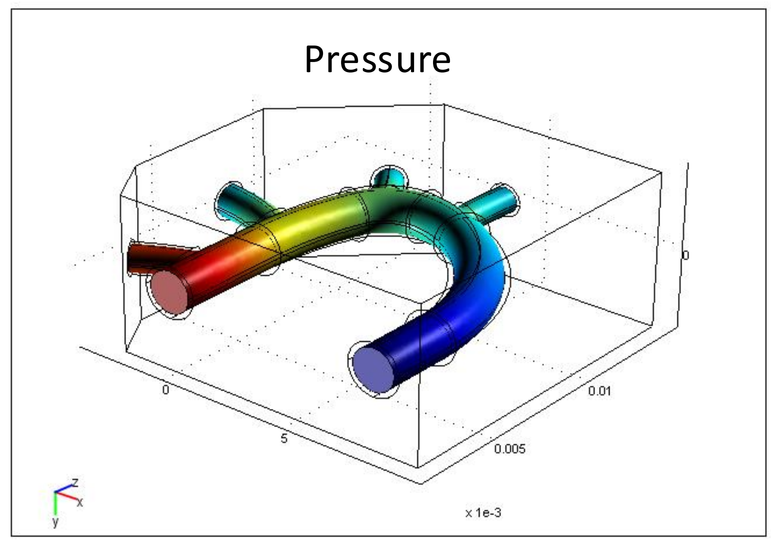


https://en.wikipedia.org/wiki/Aorta



Pressure conditions on all inlets and outlets

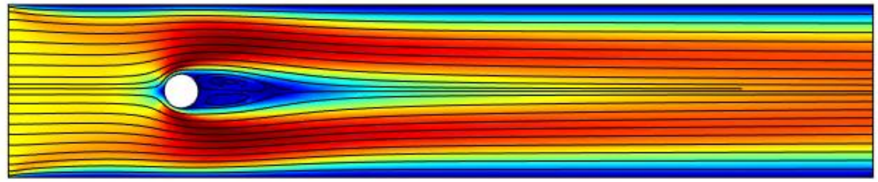




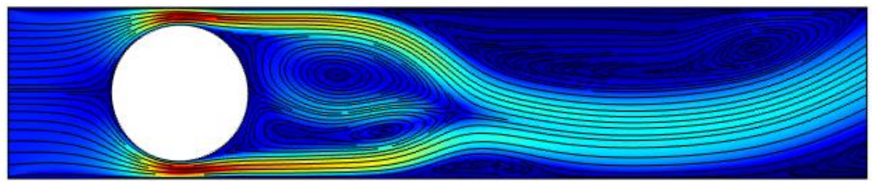
Comsol: Application Library/Structural Mechanics Module/Fluid-structure Interaction/blood_vessel

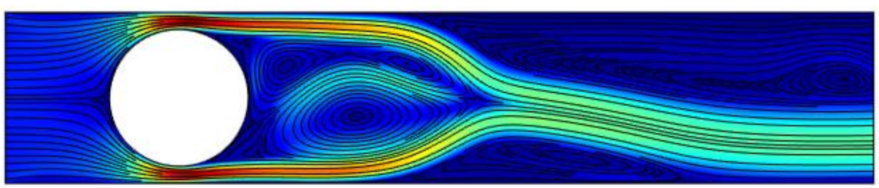
Turbulent flow

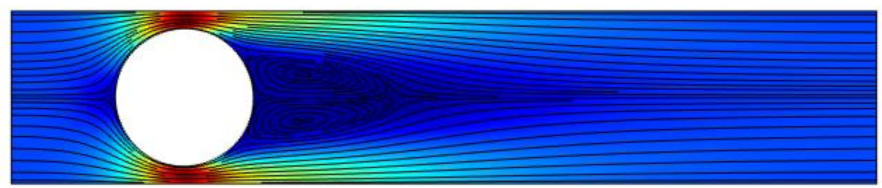
Laminar flow



Turbulent flow







Refine mesh

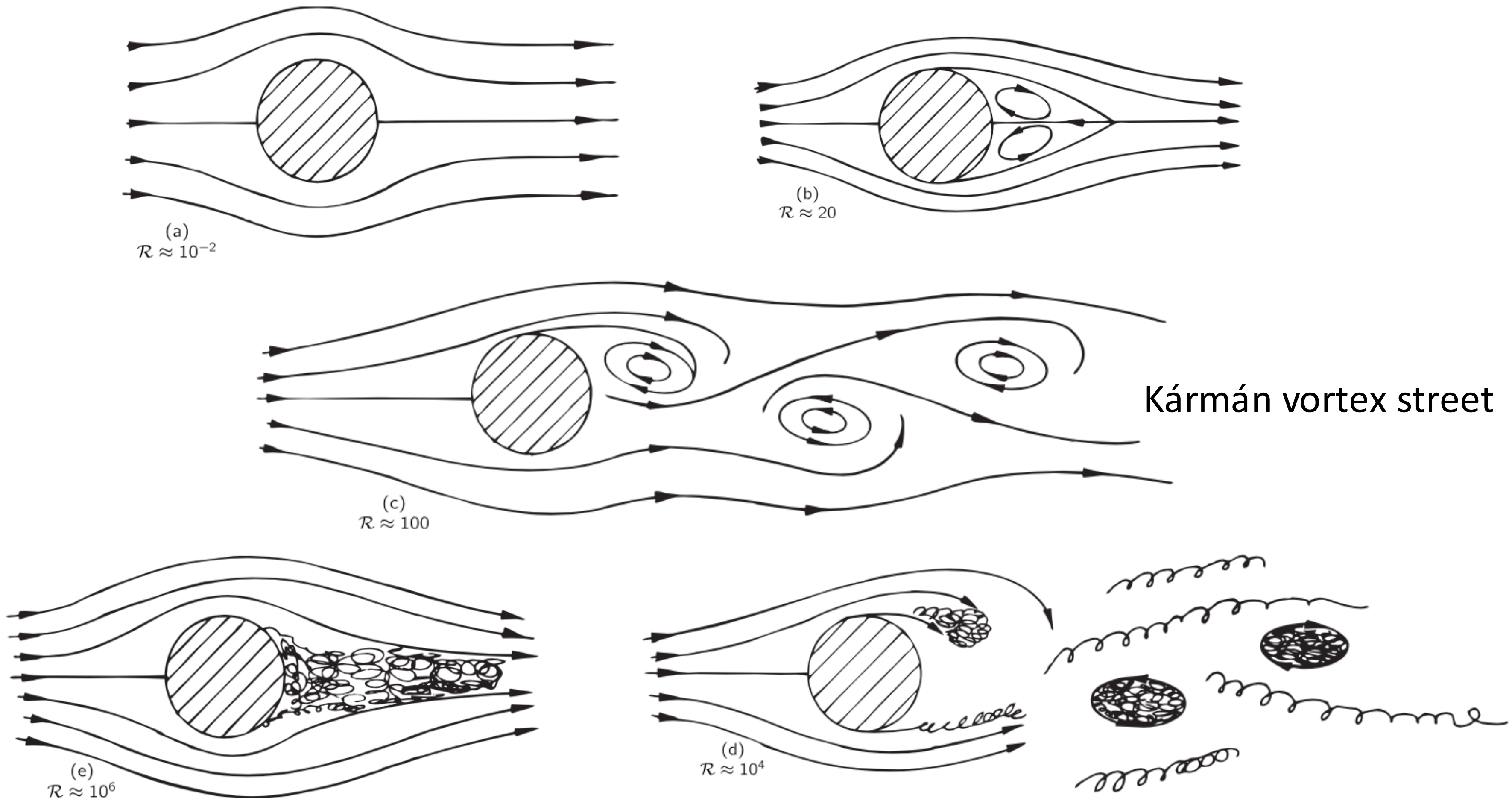
Direct Numerical Simulation (DNS)

Turbulent model

Reynolds-Averaged Navier–Stokes (RANS)

Turbulent flow

Flow past a cylinder for various Reynolds numbers



https://www.feynmanlectures.caltech.edu/II_41.html

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