APPENDIX C

Womersley's Functions of the Parameter α

Womersley's (1955a, 1957a) equation for the velocity of blood in a rigid tube contains the complex expression (eq. 5.11, p. 107):

$$\left\{1 - \frac{J_0(j^{3/2}\alpha y)}{J_0(j^{3/2}\alpha)}\right\} \qquad \textbf{C.1}$$

The numerical solution of the equation can be carried out by separating real and imaginary parts of the complex numbers (Witzig, 1914; Lambossy, 1952a), but Womersley adopted a modulus and phase form that was easier to manipulate. Following McLachlen's (1941) notation, he defined the Bessel functions as

$$J_0(j^{3/2}x) = M_0(x)\exp\,[j\theta_0(x)] \qquad \textbf{C.2}$$

so that

$$\frac{J_0(j^{3/2}\alpha y)}{J_0(j^{3/2}\alpha)} = \frac{M_0(\alpha y)}{M_0(\alpha)}\exp\,[j(\theta_0(\alpha y) - \theta_0(\alpha))]. \qquad \textbf{C.3}$$

He then defined two new terms:

$$h_0 = M_0(\alpha y)/M_0(\alpha) \qquad \textbf{C.4}$$
$$\delta_0 = \theta_0(\alpha y) - \theta_0(\alpha) \qquad \textbf{C.5}$$

so that the complex expression in equation C.1 could be compressed into

$$(1 - h_0 e^{j\delta_0}) \qquad \textbf{C.6}$$

Going on to define a modulus, M'_0, and a phase angle, ε'_0, such that

$$M'_0 = (1 - 2h_0\cos\delta_0 + h_0^2)^{1/2} \qquad \textbf{C.7}$$
$$\tan\varepsilon'_0 = \frac{h_0\sin\delta_0}{1 - h_0\cos\delta_0}, \qquad \textbf{C.8}$$

Womersley arrived at

$$\left\{1 - \frac{J_0(j^{3/2}\alpha y)}{J_0(j^{3/2}\alpha)}\right\} = M'_0 e^{j\varepsilon'_0} \qquad \textbf{C.9}$$

This equivalence provided a relatively compact form for his subsequent derivations (*e.g.*, eq. 5.16, p. 108).

In a similar way, he simplified the Bessel function part of his flow equation (eq. 5.13, p. 108), namely,

$$\left\{1 - \frac{2J_1(j^{3/2}\alpha)}{j^{3/2}\alpha J_0(j^{3/2}\alpha)]}\right\} \qquad \textbf{C.10}$$

by defining the following terms, again using the notation of equation C.2:

$$h_{10} = 2M_1(\alpha)/\alpha M_0(\alpha) \tag{C.11}$$

$$\delta_{10} = (3\pi/4) - \theta_1(\alpha) + \theta_0(\alpha) \tag{C.12}$$

$$M'_{10} = (1 - 2h_{10} \cos \delta_{10} + h_{10}^2)^{1/2} \tag{C.13}$$

$$\tan \varepsilon'_{10} = (h_{10} \sin \delta_{10})/(1 - h_{10} \cos \delta_{10}) \tag{C.14}$$

The subscript "10" distinguishes these terms from the others derived above, which depended only on Bessel functions of the J_0 type. The analogous terms for flow in a thin-walled elastic tube (M''_{10}, ε''_{10}) were derived by Womersley (1955b) in a similar way.

For some purposes he expressed results in the complex numbers

$$M'_{10}e^{j\varepsilon'_{10}} = (1 - F_{10}) \tag{C.15}$$

$$M''_{10}e^{j\varepsilon''_{10}} = (1 + NF_{10}) \tag{C.16}$$

Values of M'_{10} and ε'_{10} for α = 0–10 were published by Womersley (1957a) and are given in Table C.1. At higher values of α, these functions can be calculated from the asymptotic expansions (McDonald, 1974)

$$M'_{10} = 1 - \frac{\sqrt{2}}{\alpha} + \frac{1}{\alpha^2} \tag{C.17}$$

$$\varepsilon'_{10} = \frac{\sqrt{2}}{\alpha} + \frac{1}{\alpha^2} + \frac{19}{24(\sqrt{2})\alpha^3} \tag{C.18}$$

(The angle given by eq. C.18 is in radians). The data listed in Table C.1 for α = 12–40 were computed from these equations. Womersley (1957a) also tabulated $(1 + NF_{10})$ and the parameter $(X - jY)$ (see p. 114) for $K = +0.4$ to -10, $\alpha = 0$–0.5. These same functions are often involved in the models conceived by other investigators (Witzig, 1914; Iberall, 1950; Morgan and Kiely, 1954; Jager *et al.*, 1965; Klip *et al.*, 1967; Cox, 1967).

TABLE C.1. Womersley's functions M'_{10} and ε'_{10} for α = 0–40[a]

α	M'_{10}	ε'_{10}[b]	α	M'_{10}	ε'_{10}[b]	α	M'_{10}	ε'_{10}[b]	α	M'_{10}	ε'_{10}[b]
0.00	0.0000	90.00	2.75	.5802	39.96	5.50	.7745	16.73	8.00	.8384	11.06
.05	.0003	89.98	2.80	.5882	39.05	5.55	.7762	16.56	8.05	.8393	10.98
.10	.0012	89.90	2.85	.5959	73.16	5.60	.7780	16.39	8.10	.8402	10.91
.15	.0028	89.79	2.90	.6032	37.32	5.65	.7796	16.23	8.15	.8411	10.84
.20	.0050	89.62	2.95	.6102	36.50	5.70	.7813	16.07	8.20	.8420	10.77
.25	.0078	89.40	3.00	.6169	35.70	5.75	.7830	15.91	8.25	.8429	10.70
.30	.0112	89.14	3.05	.6233	34.93	5.80	.7846	15.76	8.30	.8437	10.63
.35	.0153	88.83	3.10	.6294	34.18	5.85	.7862	15.61	8.35	.8446	10.56
.40	.0200	88.47	3.15	.6353	33.46	5.90	.7878	15.46	8.40	.8454	10.49
.45	.0253	88.07	3.20	.6409	32.77	5.95	.7893	15.32	8.45	.8463	10.42
.50	.0312	87.61	3.25	.6463	32.09	6.00	.7909	15.18	8.50	.8471	10.36
.55	.0378	87.11	3.30	.6514	31.45	6.05	.7924	15.04	8.55	.8479	10.29
.60	.0449	86.57	3.35	.6563	30.82	6.10	.7939	14.90	8.60	.8487	10.22
.65	.0527	85.97	3.40	.6611	30.22	6.15	.7953	14.77	8.65	.8495	10.16
.70	.0610	85.33	3.45	.6656	29.64	6.20	.7968	14.63	8.70	.8503	10.10
.75	.0700	84.65	3.50	.6700	29.08	6.25	.7982	14.50	8.75	.8511	10.04
.80	.0795	83.91	3.55	.6742	28.53	6.30	.7997	14.38	8.80	.8519	9.97
.85	.0896	83.14	3.60	.6783	28.01	6.35	.8010	14.25	8.85	.8526	9.91
.90	.1003	83.32	3.65	.6822	27.51	6.40	.8024	14.13	8.90	.8534	9.85
.95	.1115	81.45	3.70	.6860	27.02	6.45	.8038	14.01	8.95	.8541	9.79
1.00	.1232	80.55	3.75	.6896	26.55	6.50	.8051	13.89	9.00	.8549	9.73
1.05	.1354	79.60	3.80	.6931	26.10	6.55	.8065	13.77	9.05	.8556	9.68
1.10	.1481	78.61	3.85	.6965	25.66	6.60	.8078	13.66	9.10	.8564	9.62
1.15	.1612	77.59	3.90	.6999	25.24	6.65	.8090	13.54	9.15	.8571	9.56
1.20	.1747	76.53	3.95	.7031	24.83	6.70	.8103	13.43	9.20	.8578	9.51
1.25	.1886	75.44	4.00	.7062	24.43	6.75	.8116	13.32	9.25	.8585	9.45
1.30	.2029	74.31	4.05	.7092	24.05	6.80	.8128	13.21	9.30	.8592	9.40
1.35	.2174	73.16	4.10	.7122	23.68	6.85	.8140	13.11	9.35	.8599	9.34
1.40	.2322	71.98	4.15	.7151	23.32	6.90	.8152	13.00	9.40	.8606	9.29
1.45	.2472	70.77	4.20	.7179	22.98	6.95	.8164	12.90	9.45	.8613	9.24
1.50	.2624	69.54	4.25	.7206	22.64	7.00	.8176	12.80	9.50	.8620	9.18
1.55	.2776	68.30	4.30	.7233	22.32	7.05	.8188	12.70	9.55	.8626	9.13
1.60	.2930	67.03	4.35	.7259	22.00	7.10	.8199	12.60	9.60	.8633	9.08
1.65	.3083	65.76	4.40	.7285	21.70	7.15	.8211	12.50	9.65	.8639	9.03
1.70	.3237	64.47	4.45	.7310	21.40	7.20	.8222	12.41	9.70	.8646	8.98
1.75	.3389	63.18	4.50	.7334	21.11	7.25	.8233	12.31	9.75	.8652	8.93
1.80	.3540	61.89	4.55	.7358	20.84	7.30	.8244	12.22	9.80	.8659	8.88
1.85	.3690	60.59	4.60	.7382	20.56	7.35	.8255	12.13	9.85	.8665	8.84
1.90	.3837	59.30	4.65	.7405	20.30	7.40	.8265	12.04	9.90	.8671	8.79
1.95	.3982	58.02	4.70	.7428	20.05	7.45	.8276	11.95	9.95	.8677	8.74
2.00	.4125	56.74	4.75	.7450	19.80	7.50	.8276	11.87	10	.8684	8.69
2.05	.4264	55.47	4.80	.7472	19.55	7.55	.8297	11.78	12	.8890	7.16
2.10	.4400	54.22	4.85	.7493	19.32	7.60	.8307	11.70	14	.9040	6.09
2.15	.4532	52.98	4.90	.7515	19.09	7.65	.8317	11.61	16	.9155	5.29
2.20	.4660	51.77	4.95	.7536	18.86	7.70	.8327	11.53	18	.9245	4.68
2.25	.4784	50.57	5.00	.7556	18.65	7.75	.8336	11.45	20	.9318	4.20
2.30	.4905	49.39	5.05	.7576	18.43	7.80	.8346	11.37	25	.9450	3.33
2.35	.5021	48.24	5.10	.7596	18.23	7.85	.8356	11.29	30	.9540	2.76
2.40	.5133	47.11	5.15	.7616	18.02	7.90	.8365	11.21	35	.9604	2.36
2.45	.5241	46.01	5.20	.7635	17.83	7.95	.8375	11.14	40	.9653	2.06
2.50	.5344	44.93	5.25	.7654	17.63						
2.55	.5444	43.88	5.30	.7673	17.44						
2.60	.5539	42.86	5.35	.7691	17.26						
2.65	.5631	41.86	5.40	.7709	17.08						
2.70	.5718	40.90	5.45	.7727	16.90						

[a] From Womersley (1957a).
[b] Values of ε'_{10} are in degrees.