2.11 Viscoelasticity

When a body is suddenly strained and then the strain is maintained constant afterward, the corresponding stresses induced in the body decrease with time. This phenomena is called *stress relaxation*, or *relaxation* for short. If the body is suddenly stressed and then the stress is maintained constant afterward, the body continues to deform, and the phenomenon is called *creep*. If the body is subjected to a cyclic loading, the stress-strain relationship in the loading process is usually somewhat different from that in the unloading process, and the phenomenon is called *hysteresis*.

The features of hysteresis, relaxation, and creep are found in many materials. Collectively, they are called features of *viscoelasticity*.

Mechanical models are often used to discuss the viscoelastic behavior of materials. In Fig. 2.11:1 are shown three mechanical models of material behavior, namely, the *Maxwell model*, the *Voigt model*, and the *Kelvin model* (also called the *standard linear solid*), all of which are composed of combinations of linear springs with spring constant μ and dashpots with coefficient of viscosity η. A *linear spring* is supposed to produce instantaneously a deformation proportional to the load. A *dashpot* is supposed to produce a velocity proportional to the load at any instant. Thus if F is the force acting in a spring and u is its extension, then $F = \mu u$. If the force F acts on a dashpot, it will produce a velocity of deflection $\dot{u}$, and $F = \eta \dot{u}$. The shock absorber

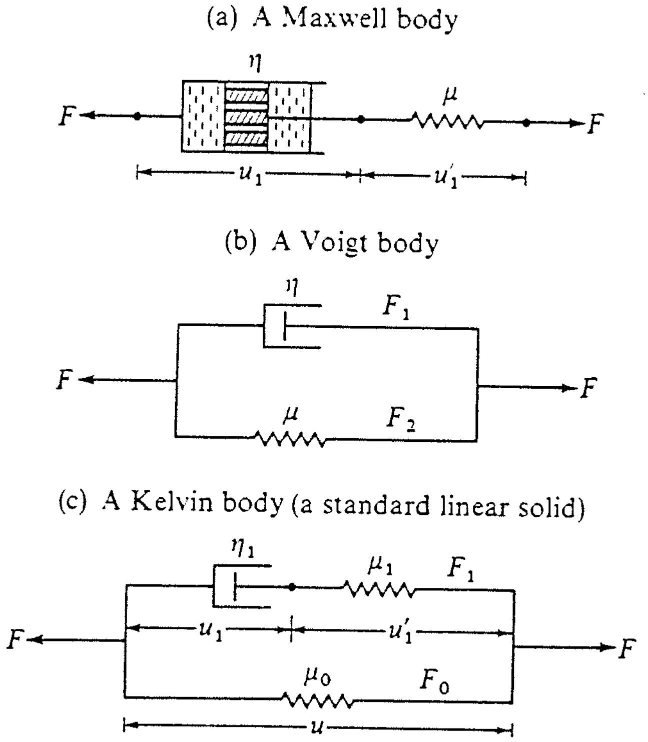

Figure 2.11:1 Three mechanical models of viscoelastic material. (a) a Maxwell body, (b) a Voigt body, and (c) a Kelvin body (a standard linear solid).

on an airplane's landing gear is an example of a dashpot. Now, in a *Maxwell model*, shown in Fig. 2.11:.1(a), the same force is transmitted from the spring to the dashpot. This force produces a displacement F/μ in the spring and a velocity F/η in the dashpot. The velocity of the spring extension is $\dot{F}/\mu$ if we denote a differentiation with respect to time by a dot. The total velocity is the sum of these two:

$$\dot{u} = \frac{\dot{F}}{\mu} + \frac{F}{\eta} \qquad \text{(Maxwell model).} \tag{1}$$

Furthermore, if the force is suddenly applied at the instant of time $t = 0$, the spring will be suddenly deformed to $u(0) = F(0)/\mu$, but the initial dashpot deflection would be zero, because there is no time to deform. Thus the initial condition for the differential equation (1) is

$$u(0) = \frac{F(0)}{\mu}. \tag{2}$$

For the *Voigt model*, the spring and the dashpot have the same displacement. If the displacement is u, the velocity is $\dot{u}$, and the spring and dashpot will produce forces μu and $\eta\dot{u}$, respectively. The total force F is therefore

$$F = \mu u + \eta\dot{u} \qquad \text{(Voigt model).} \tag{3}$$

If F is suddenly applied, the appropriate initial condition is

$$u(0) = 0. \tag{4}$$

For the Kelvin model (or standard linear model), let us break down the displacement u into u_1 of the dashpot and u_1' for the spring, whereas the total force F is the sum of the force F_0 from the spring and F_1 from the Maxwell element. Thus

$$\begin{aligned} &(a)\ u = u_1 + u_1' \qquad &&(b)\ F = F_0 + F_1. \\ &(c)\ F_0 = \mu_0 u, \qquad &&(d)\ F_1 = \eta_1\dot{u}_1 = \mu_1 u_1' \end{aligned} \tag{5}$$

From this we can verify by substitution that

$$F = \mu_0 u + \mu_1 u_1' = (\mu_0 + \mu_1)\, u - \mu_1 u_1.$$

Hence

$$F + \frac{\eta_1}{\mu_1}\dot{F} = (\mu_0 + \mu_1)u - \mu_1 u_1 + \frac{\eta_1}{\mu_1}\ (\mu_0 + \mu_1)\dot{u} - \eta_1\dot{u}_1.$$

Replacing the last term by $\mu_1 u_1'$ and using Eq. (5a), we obtain

$$F + \frac{\eta_1}{\mu_1}\dot{F} = \mu_0 u + \eta_1\left(1 + \frac{\mu_0}{\mu_1}\right)\dot{u}. \tag{6}$$

This equation can be written in the form

$$F + \tau_\varepsilon\dot{F} = E_R(u + \tau_\sigma\dot{u}), \qquad \text{(Kelvin model)} \tag{7}$$

where

$$\tau_\varepsilon = \frac{\eta_1}{\mu_1}, \qquad \tau_\sigma = \frac{\eta_1}{\mu_0}\left(1 + \frac{\mu_0}{\mu_1}\right), \qquad E_R = \mu_0. \tag{8}$$

For a suddenly applied force $F(0)$ and displacement $u(0)$, the initial condition is

$$\tau_\varepsilon F(0) = E_R \tau_\sigma u(0). \tag{9}$$

For reasons that will become clear below, the constant τ_ε is called the *relaxation time for constant strain*, whereas τ_σ is called *the relaxation time for constant stress.*

If we solve Eqs. (1), (3), and (7) for $u(t)$ when $F(t)$ is a *unit-step function* $1(t)$, the results are called *creep functions*, which represent the elongation produced by a sudden application at $t = 0$ of a constant force of magnitude unity. They are:

Maxwell solid:

$$c(t) = \left(\frac{1}{\mu} + \frac{1}{\eta}t\right)1(t), \tag{10}$$

Voigt solid:

$$c(t) = \frac{1}{\mu}(1 - e^{-(\mu/\eta)t})1(t), \tag{11}$$

Standard linear solid:

$$c(t) = \frac{1}{E_R}\left[1 - \left(1 - \frac{\tau_\varepsilon}{\tau_\sigma}\right)e^{-t/\tau_\sigma}\right]1(t), \tag{12}$$

where the unit-step function $1(t)$ is defined as [see Fig. 2.11:2(a)]

$$1(t) = \begin{cases} 1 & \text{when } t > 0, \\ \frac{1}{2} & \text{when } t = 0, \\ 0 & \text{when } t < 0. \end{cases} \tag{13}$$

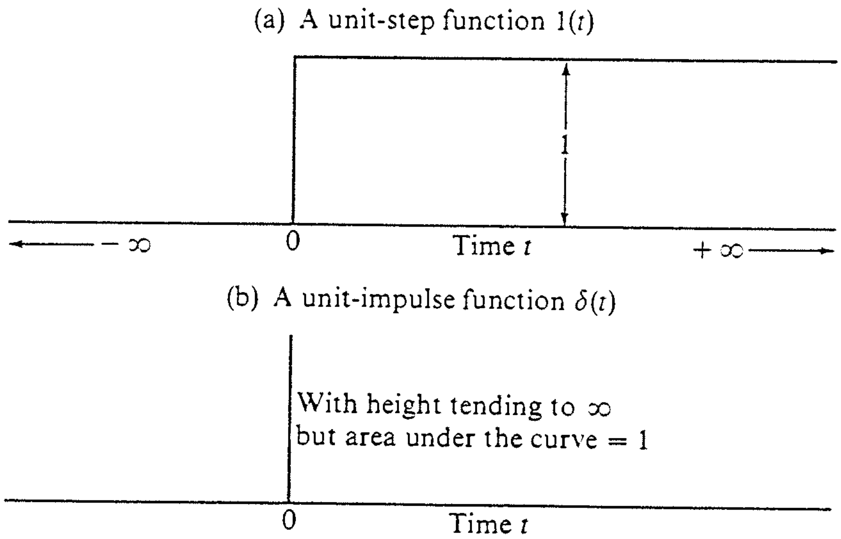

Figure 2.11:2 (a) A unit-step function $1(t)$. (b) A unit-impulse function $\delta(t)$. The central spike has a height tending to ∞ but the area under the curve remains to be unity.

A body that obeys a load-deflection relation like that given by Maxwell's model is said to be a *Maxwell solid*. Since a dashpot behaves as a piston moving in a viscous fluid, the above-named models are called models of viscoelasticity.

Interchanging the roles of F and u, we obtain the *relaxation function* as a response $F(t) = k(t)$ corresponding to an elongation $u(t) = \mathbf{1}(t)$. The relaxation function $k(t)$ is the force that must be applied in order to produce an elongation that changes at $t = 0$ from zero to unity and remains unity thereafter. They are

Maxwell solid:

$$k(t) = \mu e^{-(\mu/\eta)t}\mathbf{1}(t), \tag{14}$$

Voigt solid:

$$k(t) = \eta\delta(t) + \mu\mathbf{1}(t), \tag{15}$$

Standard linear solid:

$$k(t) = E_R\left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\varepsilon}\right)e^{-t/\tau_\varepsilon}\right]\mathbf{1}(t). \tag{16}$$

Here we have used the symbol $\delta(t)$ to indicate the *unit-impulse function or Dirac-delta function*, which is defined as a function with a singularity at the origin (see Fig. 2.11:2(b)):

$$\begin{aligned} &\delta(t) = 0 && (\text{for } t < 0, \text{ and } t > 0), \\ &\int_{-\varepsilon}^{\varepsilon} f(t)\delta(t)\,dt = f(0) && (\varepsilon > 0), \end{aligned} \tag{17}$$

where $f(t)$ is an arbitrary function, continuous at $t = 0$. These functions, $c(t)$ and $k(t)$, are illustrated in Figs. 2.11:3 and 2.11:4, respectively, for which we add the following comments.

For the Maxwell solid, the sudden application of a load induces an immediate deflection by the elastic spring, which is followed by "creep" of the dashpot. On the other hand, a sudden deformation produces an immediate reaction by the spring, which is followed by stress relaxation

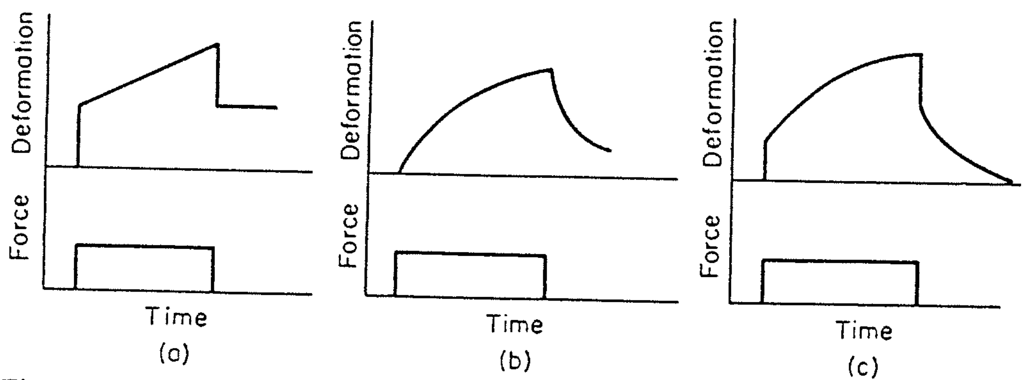

Figure 2.11:3 Creep functions of (a) a Maxwell, (b) a Voigt, and (c) a standard linear solid. A negative phase is superposed at the time of unloading.

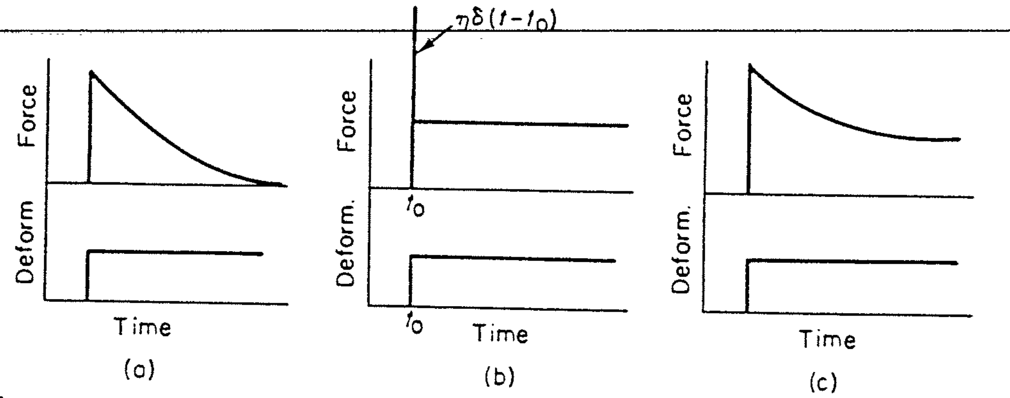

Figure 2.11:4 Relaxation functions of (a) a Maxwell, (b) a Voigt, and (c) a standard linear solid.

according to an exponential law [see Eq. (14)]. The factor η/μ, with the dimension of time, may be called the *relaxation time*: it characterizes the rate of decay of the force.

For the Voigt solid, a sudden application of force will produce no immediate deflection, because the dashpot, arranged in parallel with the spring, will not move instantaneously. Instead, as shown by Eq. (11) and Fig. 2.11:3, a deformation will be gradually built up, while the spring takes a greater and greater share of the load. The dashpot displacement relaxes exponentially. Here the ratio η/μ is again a relaxation time: it characterizes the rate of relaxation of the dashpot.

For the standard linear solid, a similar interpretation is applicable. The constant τ_ε is the time of relaxation of load under the condition of constant deflection [see Eq. (16)], whereas the constant τ_σ is the time of relaxation of deflection under the condition of constant load [see Eq. (12)]. As $t \to \infty$, the dashpot is completely relaxed, and the load-deflection relation becomes that of the springs, as is characterized by the constant E_R in Eqs. (12) and (16). Therefore, E_R is called the *relaxed elastic modulus*.

Maxwell introduced the model represented by Eq. (3), with the idea that all fluids are elastic to some extent. Lord Kelvin showed the inadequacy of the Maxwell and Voigt models in accounting for the rate of dissipation of energy in various materials subjected to cyclic loading. Kelvin's model is commonly called the standard linear model because it is the most general relationship that includes the load, the deflection, and their first (commonly called "linear") derivatives.

More general models may be built by adding more and more elements to the Kelvin model. Equivalently, we may add more and more exponential terms to the creep function or to the relaxation function.

The most general formulation under the assumption of linearity between cause and effect is due to Boltzmann (1844–1906). In the one-dimensional case, we may consider a simple bar subjected to a force $F(t)$ and elongation $u(t)$. The elongation $u(t)$ is caused by the total history of the loading up to the time t. If the function $F(t)$ is continuous and differentiable, then in a

2.12 Response of a Viscoelastic Body to Harmonic Excitation

Since biological tissues are all viscoelastic, and since one of the simplest ways to experimentally determine the viscoelastic properties is to subject the material to periodic oscillations, we shall discuss this case in greater detail.

* See Fung (1965), *Solid Mechanics*, p. 448 for details.

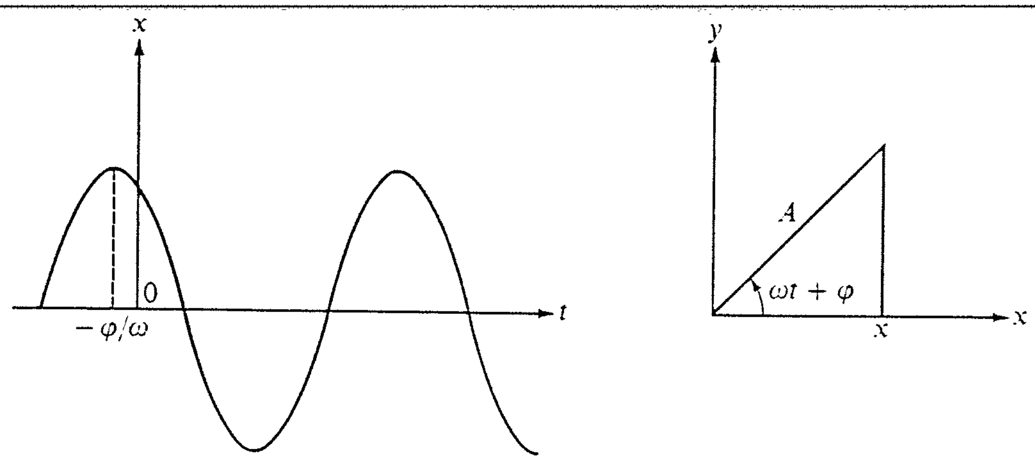

Figure 2.12:1 Complex representation of a harmonic motion.

Consider a quantity x, which varies periodically with frequency ω (radians per second) according to the rule

$$x = A\cos(\omega t + \varphi). \tag{1}$$

This is a *simple harmonic motion*; A is the *amplitude* and φ is the *phase angle*. We may consider x to be the projection of a rotating vector on the real axis (see Fig. 2.12:1). Since a vector is specified by two components, it can be represented by a complex number. For example, the vector in Fig. 2.12:1 can be specified by the components $x = A\cos(\omega t + \varphi)$ and $y = A\sin(\omega t + \varphi)$, and hence by the complex number $x + iy$. But

$$e^{i(\omega t + \varphi)} = \cos(\omega t + \varphi) + i\sin(\omega t + \varphi), \tag{2}$$

so the rotating vector can be represented by the complex number

$$x + iy = Ae^{i(\omega t + \varphi)} = Be^{i\omega t}, \tag{3}$$

where

$$B = Ae^{i\varphi}. \tag{3a}$$

Equation (1) is the real part of Eq. (3), and the latter is said to be the *complex representation* of Eq. (1). B is a complex number whose absolute value is the *amplitude*, and whose polar angle $\varphi = \arctan(\text{Im } B/\text{Rl } B)$ is the *phase angle* of the motion.

The vector representation is very convenient for "composing" several simple harmonic oscillations of the same frequency. For example, if

$$x = A_1\cos(\omega t + \varphi_1) + A_2\cos(\omega t + \varphi_2) = A\cos(\omega t + \varphi), \tag{4}$$

then x is the real part of the resultant of two vectors as shown in Fig. 2.12:2.

Now, if the force and displacement are harmonic functions of time, then we can apply complex representation. Let $u = Ue^{i\omega t}$. Then by differentiation with respect to t, we have $\dot{u} = i\omega Ue^{i\omega t} = i\omega u$. In this case a differentiation

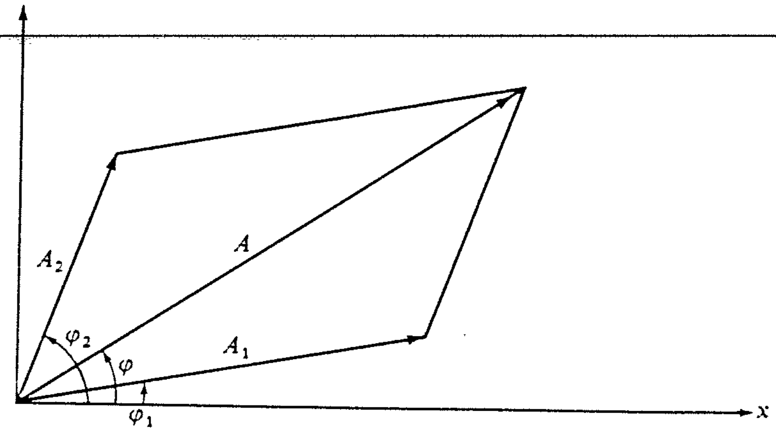

Figure 2.12:2 Vector sum of two simple harmonic motions of the same frequency.

with respect to t is equivalent to a multiplication by $i\omega$. Applying this result to Eq. (1) of Sec. 2.11, we obtain

$$i\omega u = \frac{i\omega F}{\mu} + \frac{F}{\eta}. \tag{5}$$

This can be written in the form

$$F = G(i\omega)u, \tag{6}$$

which is the same as

$$Fe^{i\omega t} = G(i\omega)ue^{i\omega t}, \tag{6a}$$

where $G(i\omega)$ is called the *complex modulus of elasticity*. In the case of the Maxwell body,

$$G(i\omega) = i\omega\left(\frac{i\omega}{\mu} + \frac{1}{\eta}\right)^{-1}. \tag{7}$$

In a similar manner, Eqs. (3), (7), (19), (20), (22), and (23) of Sec. 2.11 can all be put into the form of Eq. (6), and the complex modulus of each model can be derived.

The complex modulus of elasticity of the Kelvin body (standard linear solid), corresponding to Eq. (7) of Sec. 2.11, is

$$G(i\omega) = \frac{1 + i\omega\tau_\sigma}{1 + i\omega\tau_\varepsilon} E_R. \tag{8}$$

Writing

$$G(i\omega) = |G|e^{i\delta}, \tag{9}$$

where $|G|$ is the amplitude of the complex modulus and δ is the phase shift, we have

$$|G| = \left(\frac{1 + \omega^2\tau_\sigma^2}{1 + \omega^2\tau_\varepsilon^2}\right)^{1/2} E_R, \qquad \tan\delta = \frac{\omega(\tau_\sigma - \tau_\varepsilon)}{1 + \omega^2(\tau_\sigma\tau_\varepsilon)}. \tag{10}$$

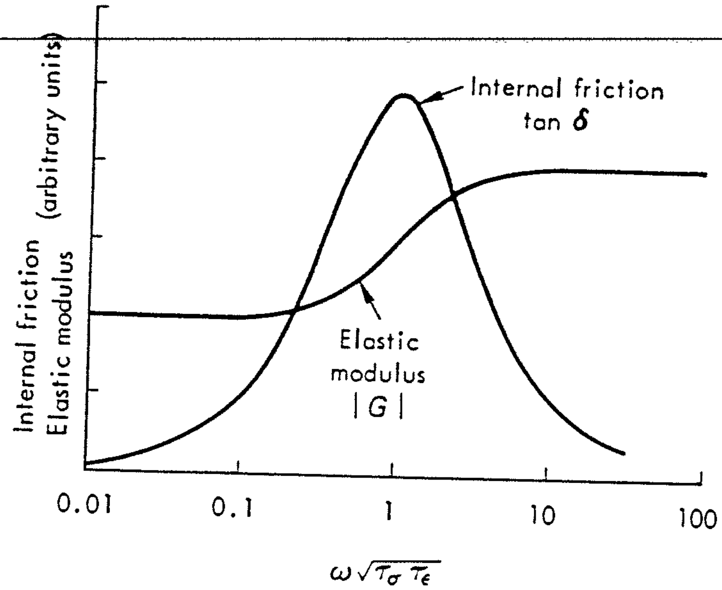

Figure 2.12:3 The dynamic modulus of elasticity $|G|$ and the internal damping $\tan\delta$ plotted as a function of the logarithm of frequency ω for a standard linear solid.

The quantity $\tan\delta$ is a measure of "internal friction." When $|G|$ and $\tan\delta$ are plotted against the logarithm of ω, curves as shown in Fig. 2.12:3 are obtained. The internal friction reaches a peak when the frequency ω is equal to $(\tau_\sigma\tau_\varepsilon)^{-1/2}$. Correspondingly, the elastic modulus $|G|$ has the fastest rise for frequencies in the neighborhood of $(\tau_\sigma\tau_\varepsilon)^{-1/2}$.