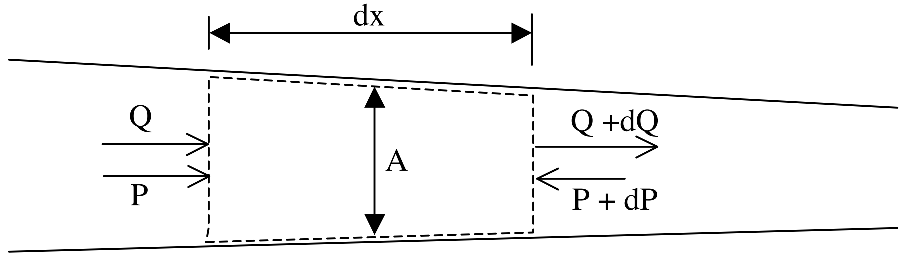
One-dimensional (1-D) flow equations



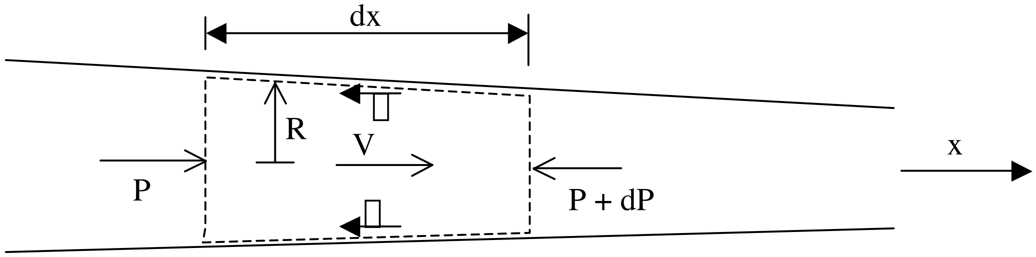
1. Conservation of mass

$$Q_{in} - Q_{out} = \frac{dV}{dt} \Rightarrow \; Q - (Q + dQ) = \frac{d(dx \cdot A)}{dt} = dx\,\frac{\partial A}{\partial t}$$

-> $\boxed{\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0}$ (1)

2. Conservation of momentum

Assumption: no radial velocity component (v << u)



Newton's law: $\sum F_x = m \cdot a_x$

$$\left[P - (P + dP)\right] \cdot A - \tau \cdot (2\pi R \cdot dx) = (\rho \cdot A \cdot dx)\frac{\partial V}{\partial t}$$

$$-\frac{\partial P}{\partial x} \cdot A - \tau \cdot 2\pi R = \rho \cdot \frac{\partial Q}{\partial t}$$

Note: Q = A V and assuming that $d(A \cdot V) = dA \cdot V + A \cdot dV \approx A \cdot dV$

Finally,

$\boxed{\frac{\partial Q}{\partial t} = -\frac{A}{\rho}\frac{\partial P}{\partial x} - \tau \cdot \frac{2\pi R}{\rho}}$ (2)

We have neglected convective acceleration (tapering) effects. If included:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\alpha\frac{Q^2}{A}) = -\frac{A}{\rho}\frac{\partial P}{\partial x} - \tau \cdot \frac{2\pi R}{\rho}$$

where $\alpha = \frac{1}{A}\int \frac{u^2}{V^2} dA$ is a velocity profile shape factor. The shape factor varies between the value of $\alpha = 1$ for a flat profile to the value of $\alpha = 4/3$ for a parabolic profile. One of the main problems of the 1-D flow equations is that both α and $\tau = \mu \left.\frac{\partial u}{\partial r}\right|_{r=R}$ depend on the velocity profile which is in principle not known. The usual approximations are $\alpha = 1$ (flat profile) and $\tau = \frac{4\mu Q}{\pi R^3}$ (Poiseuille's law).

The continuity and momentum equations describe pressure and flow wave propagation in an arterial segment. They contain three variables: pressure (P), flow (Q) and area (A). In order to close the system of equations we need one more equation, which is usually given in terms of a constituitive equation describing the relationship between pressure and cross-sectional area: A = f(p). Typically, this relationship for an artery is nonlinear as shown in the following figure

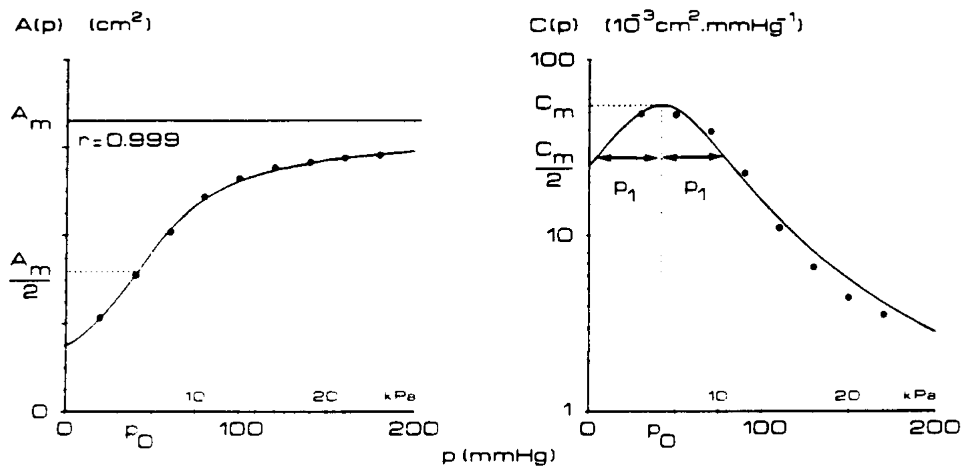


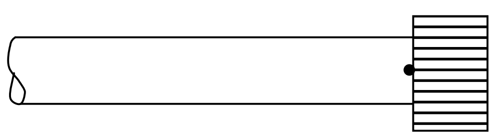
Figure 1. Area-pressure, (A(p), left) and compliance-pressure ($C_A(p)$, right) for the human aorta and for pressures of 0 tot 200 mmHg (after Langewouters 1984). Compliance is defined as the slope of the area-pressure curve.

Equations 1 and 2 together with a constitutive equation between area (A) and pressure (p) form a closed system of nonlinear first order differential equations, which can be solved using standard numerical schemes (i.e., method of characteristics or finite difference schemes). These equations hold for arterial flow in an arterial segment. One needs, however, to specify the boundary conditions at the beginning of the arterial network (typically the ascending aorta), at branching points, as well as at termination sites. These boundary conditions are usually as follows:

Upstream boundary condition: Specify either the (aortic) pressure or flow waveform. Most often the aortic flow waveform is used as the upstream boundary condition (Figure 2).

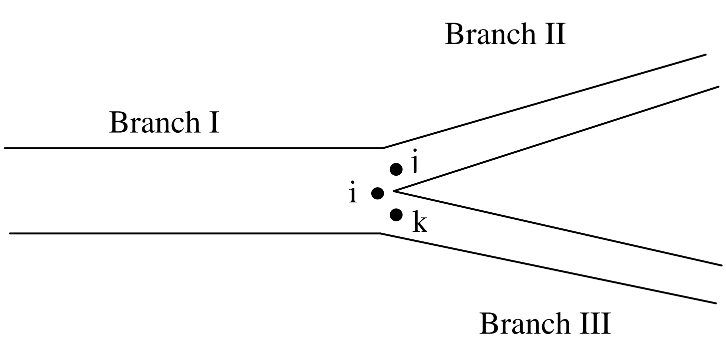
Figure 2. Aortic flow in a young subject as a function of heart cycle

Downstream boundary condition: Specify either pressure or flow waveform or the terminal (distal) impedance (Z_T). The terminal impedance model of vasculature beyond the terminal point and characterizes the relation between pressure and flow at the terminal point.



$$Z_T = \frac{P}{Q}$$

Boundary condition at bifurcation: Most commonly apply a simple continuity in pressure and flow, such as described below. Indices i, j and k denote the last node of mother branch I and the first node of daughter branch II and III, respectively.



$$p_i = p_j = p_k$$
$$Q_i = Q_j + Q_k$$