Series 6 – Solutions

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Chapter 10: Elasticity

Exercise 10.1 - Solution

The first objective of this problem is to show that, even for a Hookean material, the pressure-radius relationship is non-linear. We assume that the wall of the tube is thin and incompressible, and that the deformations are small.

From the Law of Laplace,

$$T = p \cdot r \tag{1}$$

where *T* is the tension, *p* is the pressure and *r* is the radius. Taking the differential gives:

$$\mathrm{d}T = p \cdot \mathrm{d}r + r \cdot \mathrm{d}p \tag{2}$$

so that changes in tension depend on the pressure and radius as well as the changes in these quantities.

From the behavior of a Hookean material under small deformations:

$$\mathrm{d}\sigma = E_0 \cdot \mathrm{d}\varepsilon \tag{3}$$

where σ is the stress, E_0 is Young's modulus and ε is the strain. For a cylindrical geometry, the strain is the change in circumference divided by the circumference, or

$$\mathrm{d}\varepsilon = \frac{2\pi \cdot \mathrm{d}r}{2\pi \cdot r} = \frac{\mathrm{d}r}{r} \tag{4}$$

The last piece of information is the relationship between tension and stress:

$$\sigma = \frac{T}{h} \tag{5}$$

where *h* is the wall thickness. Taking the differential gives:

$$\mathrm{d}\sigma = \frac{1}{h}\mathrm{d}T - \frac{T}{h^2}\mathrm{d}h \tag{6}$$

Since the wall is assumed incompressible we can apply the incompressibility condition:

$$r \cdot h = r_0 \cdot h_0 = \mathrm{ct} \tag{7}$$

where r_0 and h_0 are the radius and thickness, respectively, at zero pressure. In differential form we can write:

$$h \cdot \mathrm{d}r + r \cdot \mathrm{d}h = 0 \tag{8}$$

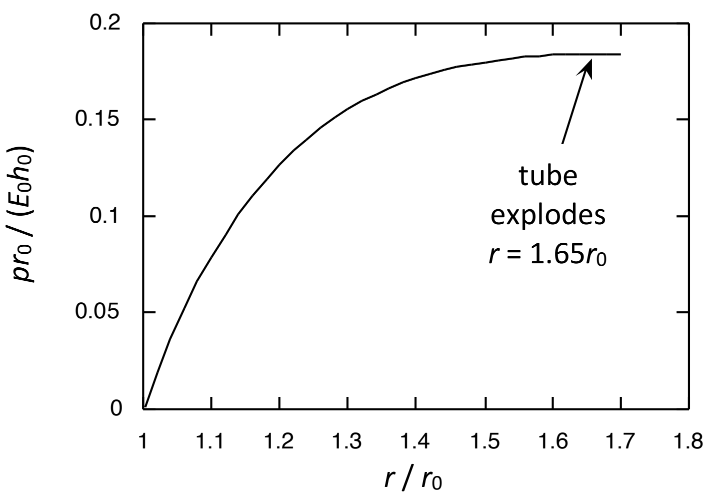
Using equations (2), (3), (4), (6), (7) and (8) we can write the following differential equation relating the radius and pressure:

$$\frac{\mathrm{d}p}{\mathrm{d}r} + \frac{2p}{r} = \frac{E_0 h_0 r_0}{r^3} \tag{9}$$

Solving this first order differential equation and applying the boundary condition $r=r_0$ at $p=0$ we obtain the solution:

$$p(r) = \frac{E_0 h_0 r_0}{r^2} \ln \frac{r}{r_0} \tag{10}$$

Graphically, the relationship looks like this:



Note that at $r/r_0 = 1.65$ the tube explodes (slope becomes zero).

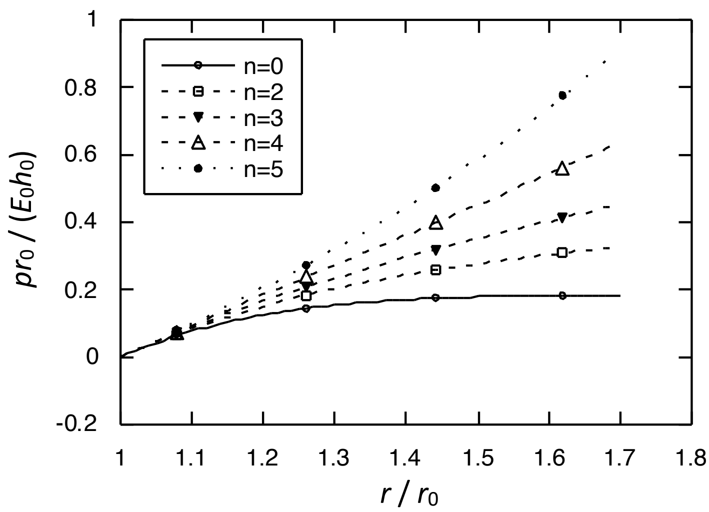
For the second part, the goal is to propose a radius dependent Young's modulus that yields a pressure - radius curve similar to that found for an artery. Due to the collagen in the arterial wall, the vessel becomes harder to distend with increasing pressure. One possible radius dependent Young's modulus which would satisfy this behaviour is:

$$E = E_0 \left(\frac{r}{r_0} \right)^n \tag{11}$$

where $n>0$. Using an approach similar to that outlined above, the pressure - radius relationship is:

$$p(r) = \frac{E_0 h_0 r_0}{r^2} \frac{1}{n} \left[\left(\frac{r}{r_0} \right)^n - 1 \right] \tag{12}$$

The results for *n* = 0 (previous case) and *n* = 2, 3, 4 and 5 are shown below:



Clearly, to obtain a pressure-stiffening (d*p*/d*r*>0) relationship like in real arteries we have to use a relationship with *n*>3.

Exercise 10.2 – Solution

If we had a blood vessel of constant radius, *r*, Poiseuille's law would give us the expression for the flow, *Q*:

$$Q = \frac{\pi r^4}{8\mu} \frac{\mathrm{d}p}{\mathrm{d}z} \tag{1}$$

where *z* is the axis of symmetry of the vessel. We now have to find the relationship between *r*(*z*) and *p*(*z*) when the arterial wall provokes conicity.

Law of Laplace:

$$\sigma_\theta = \frac{p(z)r(z)}{h} \tag{2}$$

where *h* is the wall thickness. Law of Hooke:

$$\sigma_\theta = \varepsilon_0 E \tag{3}$$

where ε_0 is the circumferential strain given by:

$$\varepsilon_0 = \frac{r(z)-r_0}{r_0} \tag{4}$$

and r_0 is a reference value for the radius, let's say $r(z=0) = r_0$. From Eqs. (2)-(4) we obtain:

$$\frac{r(z)-r_0}{r_0} = \frac{p(z)r(z)}{E\cdot h} \Rightarrow r(z) = r_0\left[1-\frac{r_0 p(z)}{E\cdot h}\right]^{-1} \tag{5}$$

If we substitute the expression for the radius from Eq. (5) into Eq. (1) we have:

$$\left[1-\frac{r_0\cdot p(z)}{E\cdot h}\right]^{-4} \mathrm{d}p = \frac{8\mu}{\pi r_0^4} Q\mathrm{d}z \tag{6}$$

Integration of Eq. 6 along vessel length, *L*, yields:

$$\frac{E\cdot h}{3r_0}\left\{\left[1-\frac{r_0\cdot p(L)}{E\cdot h}\right]^{-3} - \left[1-\frac{r_0\cdot p(0)}{E\cdot h}\right]^{-3}\right\} = \frac{8\mu}{\pi r_0^4} QL \tag{7}$$

Hence the flow, *Q*, is not a linear function of the pressure gradient any more.

Exercise 10.3 – Solution

The vascular resistance of a Poiseuille's flow is given by:

$$R_v^{rigid} = \frac{\Delta p}{Q} = \frac{128\mu L}{\pi D^4} \tag{1}$$

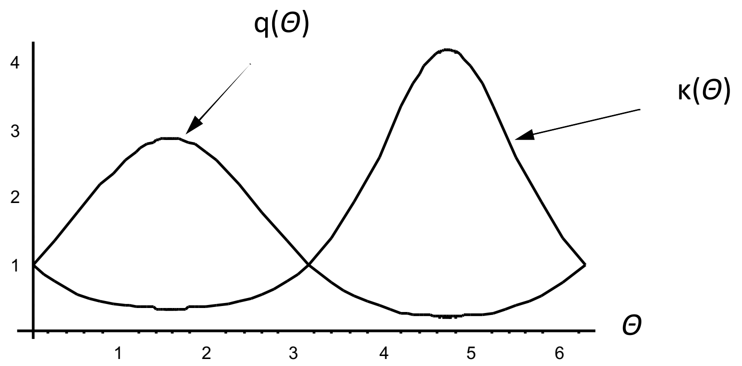
In case the diameter oscillates (vasomotion):

$$R_v^{oscill} = \frac{128\mu L}{\pi\left(D+\delta\sin\Theta\right)^4} \tag{2}$$

So the ratio κ (*Θ*) is equal to:

$$\kappa(\Theta) = \frac{\mathrm{R_v^{oscill}}}{\mathrm{R_v^{rigid}}} = (1+\alpha\sin\Theta)^{-4} \tag{3}$$

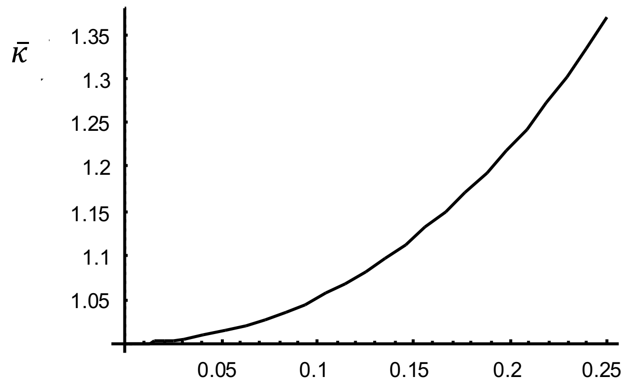
where $\alpha = \delta/D$. The flow is obtained by inversing Eq. (3). The following is graphically obtained:



The mean ratio, $\bar{\kappa}$, is calculated by integrating Eq. (3) over a cardiac cycle:

$$\bar{\kappa} = \frac{1}{2\pi}\int_0^{2\pi}(1+\alpha\sin\Theta)^{-4}d\Theta = \frac{(2+3\alpha^2)\sqrt{1-\alpha^2}}{2(\alpha^2-1)^4} \tag{4}$$

The figure below depicts the variation of $\bar{\kappa}$ for different values of α. We observe that the mean ratio increases due to vasomotion. The mean flow increases as well. This paradox is due to the particular expression of the resistance as a function of Θ (previous graph) and the fact that flow varies inversely proportionally to the resistance.



Attention: The results were obtained by using Poiseuille's law for the pulsatile flow.