Series 5 – Solutions

Prof: Nikos Stergiopulos
TA: Sokratis Anagnostopoulos

Chapter 8: Oscillatory Flow Theory

Exercise 8.1 - Solution

a) The pressure gradient (in N/m^3) is:

$$\frac{p_1 - p_2}{L} = 5 * 10^3 + 10^3 \cos(\omega t) = p'_o + M * \cos(\omega t + \varphi) \tag{1}$$

where p'_o = 5 ∗ 10^3 N/m^3, M = 10^3 N/m^3, ω = 16 rad/s and φ = 0 rad. The instantaneous flow, $Q(t)$, is given by:

$$Q(t) = Q_o + \frac{\pi R^4}{\mu} M \frac{M'_{10}}{\alpha^2} \sin(\omega t + \varphi + \varepsilon'_{10}) \tag{2}$$

where R = 2 mm. The Womersley parameter, α, is:

$$\alpha = R\sqrt{\frac{\omega}{\nu}} = R\sqrt{\frac{\omega\rho}{\mu}} = 0.002\sqrt{\frac{16*1000}{0.004}} = 4 \tag{3}$$

From Table C.1 (Womersley's functions M'_{10} and ε'_{10} for α = 0–40) we obtain for α = 4:

$$M'_{10} = 0.7062 \Rightarrow \frac{M'_{10}}{\alpha^2} = 0.0441 \quad , \quad \varepsilon'_{10} = 24.43^\circ = 0.426 \text{ rad} \tag{4}$$

Q_o is the steady flow component corresponding to the steady pressure gradient component, p'_o. It can be calculated using Poiseuille's law:

$$Q_o = \frac{\pi R^4}{8\mu} p'_o \tag{5}$$

Substitution into Eq. (2) yields:

$$Q(t) = \frac{\pi * 0.002^4}{8 * 0.004} 5 * 10^3 + \frac{\pi * 0.002^4}{0.004} 10^3 * 0.0441 \sin(16t + 0 + 0.426) \Leftrightarrow$$

$$\Leftrightarrow Q(t) = 7.85 * 10^{-6} + 0.554 * 10^{-6} \sin(16t + 0.426) \quad [\text{m}^3/\text{s}]$$

$$\text{or} \quad Q(t) = 7.85 + 0.554 \sin(16t + 0.426) \quad [\text{cm}^3/\text{s}] \tag{6}$$

b) The mean velocity, U_m, based on the steady flow component, Q_o, is:

$$U_m = \frac{Q_o}{\pi R^2} = \frac{7.85 * 10^{-6}}{\pi * 0.002^2} = 0.625 \text{ m/s} \tag{7}$$

The Reynolds number, *Re_m*, that corresponds to mean velocity, U_m, is equal to:

$$Re_m = \frac{U_m D}{\nu} = \frac{\rho U_m (2R)}{\mu} = \frac{1000 * 0.625 * 0.004}{0.004} = 625 \tag{8}$$

Exercise 8.2 – Solution

The Womersley parameter, α, for the highest significant frequency, 4 Hz, is equal to:

$$\alpha = R\sqrt{\frac{\omega}{\nu}} = R\sqrt{\frac{\omega\rho}{\mu}} = \frac{0.5}{2 * 1000}\sqrt{\frac{(2 * \pi * 4) * 1050}{0.004}} = 0.642 \ll 1 \tag{1}$$

Womersley parameter decreases when frequency decreases. So, for frequencies less than 4 Hz, parameter α will be less than 0.642. Hence the flow is quasi-steady (i.e. quasi-Poiseuille) and approximated by:

$$Q(t) = \frac{\pi R^4}{8\mu} f(t) = \frac{\pi * \left(\frac{0.5}{2 * 1000}\right)^4}{8 * 0.004} f(t) = 3.83 * 10^{-4} f(t) \quad [\text{mm}^3/\text{s}] \tag{2}$$

where the pressure gradient, f(t), is expressed in N/m^3.