Series 4 – Solutions

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Chapter 4: Dimensional Analysis

Exercise 4.1 – Solution

The stenotic cardiac valve is shown schematically in Fig. 1.

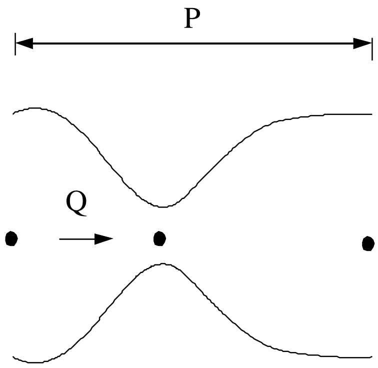

Figure 1. Schema of a stenotic cardiac valve

Let us assume that the pressure drop, ΔP [FL^{-2}], between points 1 and 3 is a function of the blood flow, Q [L^3T^{-1}], the stenosis area (at point 2), A_2 [L^2], and the blood density, ρ [ML^{-3} = FT^2L^{-4}]. The letters in brackets denote the units of each variable (F = force, L = length, T = time, M = mass). We assume that inertial effects (high acceleration through the stenosis) dominate and we can neglect viscous effects, so that viscosity does not come into play. This means that the pressure drop, ΔP, is in general given by:

$$\Delta P = \mathrm{f}(Q, A_2, \rho) \tag{1}$$

By virtue of the Buckingham Π-theorem, Eq. 1 can be written in dimensionless form using 4 (# of variables) - 3 (# of dimensions) = 1 dimensionless Π-term. Since there is only one Π-term, this can be only equal to a constant. So we can write:

$$\frac{\Delta P}{\rho\left(\frac{Q}{A_2}\right)^2} = \mathrm{c}_1 \Leftrightarrow \Delta P = \mathrm{c}_1 \rho \left(\frac{Q}{A_2}\right)^2 \tag{2}$$

or, solving for the stenotic area:

$$A_2 = \mathrm{c}_2 Q \sqrt{\frac{\rho}{\Delta P}} \tag{3}$$

where c_2 is a constant to be determined experimentally. Eq. 3 is called the Gorlin equation, and it was actually used in clinical practice to estimate the stenotic area, A_2. Nowadays, clinicians use ultrasonic and other imaging techniques for direct valve area assessment.

We could also apply fluid mechanics principle to the problem. We can, for example, apply the Bernoulli equation for the converging part of the tube, namely between points 1 and 2:

$$P_1 + \frac{\rho}{2}V_1^2 = P_2 + \frac{\rho}{2}V_2^2 \tag{4}$$

where V is the average velocity of the fluid. By continuity, $A_1V_1 = A_2V_2 = Q$. So, Eq. 4 can be rewritten as:

$$P_1 - P_2 = \frac{\rho}{2}Q^2\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right) \cong \frac{\rho}{2}\frac{Q^2}{A_2^2} \tag{5}$$

because A_1>>A_2. From point 2 to 3 there is only little pressure recovery, because most of the kinetic energy is dissipated in turbulence. So we may write (c_e: expansion loss coefficient):

$$\Delta P = P_1 - P_3 = c_e\left(P_1 - P_2\right) = c_e\frac{\rho}{2}\frac{Q^2}{A_2^2} \tag{6}$$

Eq. 6 is essentially the same as Eq. 2, with $c_1 = c_e / 2$.

Exercise 4.2 – Solution
a) In terms of units, the variables entering the problem can be written:

p ~ FL^{-2}
x ~ L
α [rad], i.e. dimensionless, i.e. Π-term
D ~ L
d ~ L
U ~ LT^{-1}
μ ~ $FL^{-2}T$
ρ ~ $FL^{-4}T^2$
P_a ~ FL^{-2}

$p = p(x, \alpha, D, d, U, \mu, \rho, P_a)$

9 variables – 3 basic dimensions = 6 Π-terms

The 6 Π-terms can be formed by inspection:

$\Pi_1 = p/P_a$
$\Pi_2 = x/D$

$\Pi_3 = d/D$
$\Pi_4 = \alpha$
$\Pi_5 = DP_a/(U\mu)$
$\Pi_6 = D\rho U/\mu$ = Reynolds number

We now have the equation $\Pi_1 = \Phi(\Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6)$ to solve.

b) The fluid (blood) does not change, thus properties μ and ρ remain the same. Using Π_6 (Reynolds number) we have:

$\Pi_6 = \text{const} \Leftrightarrow D\rho U/\mu = D'\rho' U'/\mu'$

Substituting $\rho = \rho'$, $\mu = \mu'$, U' = 5cm/s and $D = 10D'$, results in $U = U'/10$ = 5mm/s.

Exercise 4.3 – Solution
In terms of units, the variables entering the problem can be written:

$P_{dia} \sim FL^{-2}$
$Q \sim L^3T^{-1}$
$R = P/Q \sim FL^{-2} / (L^3T^{-1}) = FL^{-5}T$
$C = dV/dp \sim L^3 / (FL^{-2}) = F^{-1}L^5$
$v \sim T^{-1}$

$P_{dia} = F(Q,R,C,v)$

5 variables – 3 basic dimensions = 2 Π-terms

The 2 Π-terms can be formed by inspection: $\Pi_1 = P_{dia}/(QR)$ and $\Pi_2 = RCv$

We now have the equation $\Pi_1 = \Phi(\Pi_2)$ to solve.