Series 3 – Solutions

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Exercise 2.3: Solution

1. First let us consider a single arterial segment, as shown in Fig. 1 below.

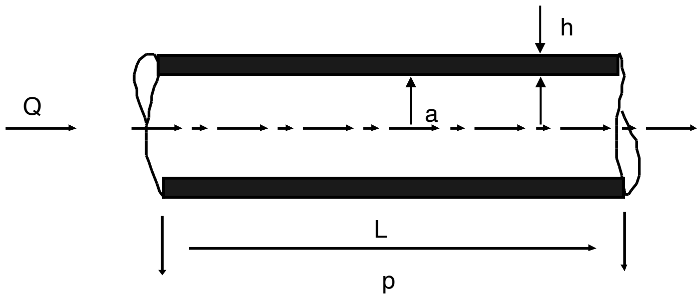

Figure 1. Parameters for a single arterial segment.

Assuming that the metabolic rate is proportional to the wall mass, it follows that the cost function P_m for the metabolic part will be

$$P_m = k_1 (2\pi a L h) \tag{1}$$

where h is the wall thickness and therefore the term within the parenthesis is the wall volume. Assuming that the wall thickness is proportional to the radius a, the total cost function P is then given by

$$P = Q\Delta p + k\pi a^2 L \tag{2}$$

The first part of the total cost function is the energy loss due to flow. If we assume that the law of Poiseuille applies, the cost function is finally given by

$$P = \frac{8\mu L}{\pi a^4} Q^2 + k\pi a^2 L \tag{3}$$

We look now for the optimum value of the radius a, that is the value that minimizes, for a given flow, the cost function. The condition is

$$\frac{\partial P}{\partial a} = -\frac{32\mu L}{\pi a^5} Q^2 + 2k\pi a L = 0 \tag{4}$$

which yields the <u>optimum radius</u>

$$a = \sqrt[6]{\frac{16\mu}{\pi^2 k}} \sqrt[3]{Q} \quad \text{or} \quad Q^2 = \frac{a^6 k \pi^2}{16\mu} \tag{5}$$

Hence, substituting into Eq. 3, we obtain the minimum cost function

$$P_{\min} = \frac{3\pi}{2} kL\mathrm{a}^2 \tag{6}$$

2. Now let us consider the bifurcation problem as given in Fig. 2 that follows.

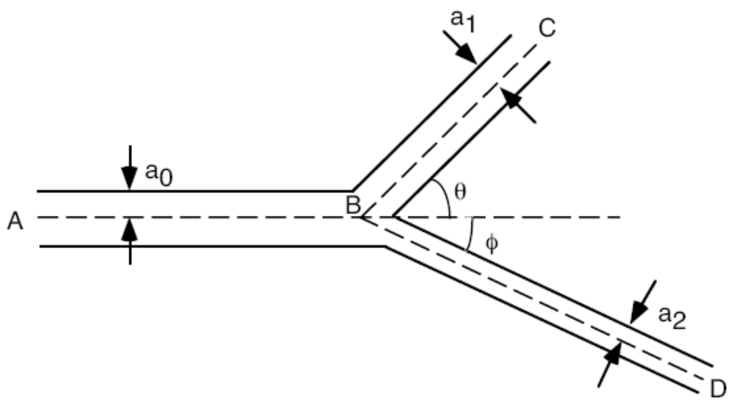

Figure 2. Parameters for the bifurcation problem.

We assume that the flows of the three segments (Q_0, Q_1, Q_2) are given and the end points of the three segments (A, C, D) are fixed. Therefore, we seek the optimum position for the intersection point B, which will obviously lie on the ACD plane. We note that the cost functions of the three vessels are additive, so using the results of Question 1 we obtain the minimum cost function of the bifurcation:

$$P = \frac{3\pi k}{2}\left(\mathrm{a}_0^2 L_0 + \mathrm{a}_1^2 L_1 + \mathrm{a}_2^2 L_2\right) \tag{7}$$

Since Eq. 5 relates the radii a_0, a_1 and a_2 to the fixed flows Q_0, Q_1 and Q_2, respectively, the only parameters that change are the individual lengths L_0, L_1 and L_2 of each branch. If we assume that the position B is optimum, then any infinitesimally small change in the position of B will produce no change in the value of the cost function, hence:

$$\Delta P = \frac{3\pi k}{2}\left(\mathrm{a}_0^2 \Delta L_0 + \mathrm{a}_1^2 \Delta L_1 + \mathrm{a}_2^2 \Delta L_2\right) = 0 \tag{8}$$

Referring to Fig. 3, we consider a small movement of point B to B' in the direction of AB.

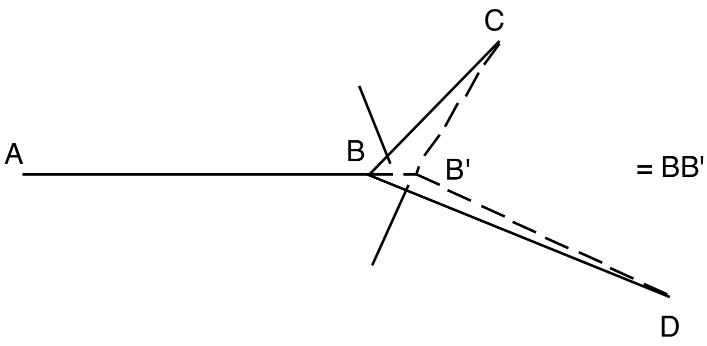

Figure 3. Variation of individual artery lengths

The variation of each segment length will be $\Delta L_0 = \delta$, $\Delta L_1 = -\delta\cdot\cos\theta$ and $\Delta L_2 = -\delta\cdot\cos\phi$. So substituting into Eq. 8 the following condition is obtained:

$$a_0^2 - a_1^2\cos\theta - a_2^2\cos\phi = 0 \tag{9}$$

If we now move point B to B' in the direction of CB (see Fig. 4), we obtain $\Delta L_0 = -\delta\cdot\cos\theta$, $\Delta L_1 = \delta$ and $\Delta L_2 = -\delta\cdot\cos(\theta+\phi)$, which results in the following relationship:

$$-a_0^2\cos\theta + a_1^2 + a_2^2\cos(\theta+\phi) = 0 \tag{10}$$

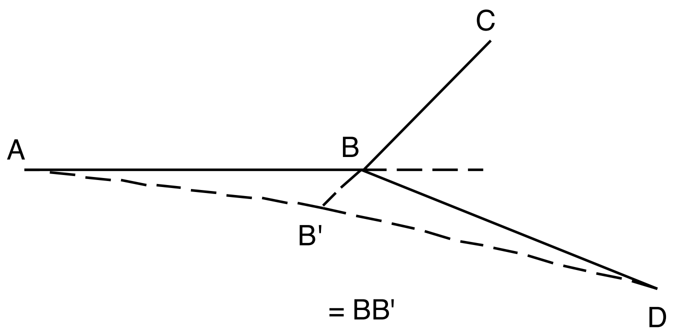

Figure 4. Variation of individual artery lengths

To obtain enough mathematical conditions, we move for a third time point B in the direction of DB, so that $\Delta L_0 = -\delta\cdot\cos\phi$, $\Delta L_1 = \delta\cdot\cos(\theta+\phi)$ and $\Delta L_2 = \delta$, yielding the following equation:

$$-a_0^2\cos\phi + a_1^2\cos(\theta+\phi) + a_2^2 = 0 \tag{11}$$

The system of Eqs. 9-11 can be solved to give the values of the angles θ and ϕ:

$$\cos\theta = \frac{a_0^4 + a_1^4 - a_2^4}{2a_0^2 a_1^2} \tag{12}$$

$$\cos\phi = \frac{a_0^4 - a_1^4 + a_2^4}{2a_0^2 a_2^2} \tag{13}$$

The mass conservation law requires that $Q_0 = Q_1 + Q_2$, which using Eq. 5 implies that

$$a_0^3 = a_1^3 + a_2^3 \tag{14}$$

Thus, Eqs. 12 and 13 can be rewritten to express θ and ϕ as functions of radii a_0 and a_1 only:

$$\cos\theta = \frac{a_0^4 + a_1^4 - \sqrt[3]{\left(a_0^3 - a_1^3\right)^4}}{2a_0^2 a_1^2} \tag{15}$$

$$\cos\phi = \frac{a_0^4 - a_1^4 + \sqrt[3]{\left(a_0^3 - a_1^3\right)^4}}{2a_0^2 \sqrt[3]{\left(a_0^3 - a_1^3\right)^2}} \tag{16}$$

3. If the radii of the two daughter branches are equal ($a_1 = a_2$), it follows from Eqs. 12 and 13 that the angles θ and ϕ are equal. From Eq. 14 we obtain

$$2a_1^3 = a_0^3 \Rightarrow a_1 = \sqrt[3]{0.5}a_0 = 0.794a_0 \tag{17}$$

If for all generations of arteries the bifurcations are of the same type so that the above condition applies for each bifurcation, the radius a_n at the n^{th} generation of arteries will be

$$a_n = 0.794^n a_0 \tag{18}$$

Based on that, to go from the aorta with $a_0 = 1.3$ cm to the capillaries with $a_n = 5\cdot10^{-4}$ cm, the necessary number of generations of arteries is

$$n = \frac{\ln\frac{a_n}{a_0}}{\ln 0.794} = 34 \tag{19}$$

Since in each generation the number of vessels doubles, 34 generations would yield $2^{34} = 1.7\cdot10^{10}$ capillaries. This number is not far from the estimated number of capillaries ($5\cdot10^9$) which was given in the class notes (slide 5, Introduction to CV system).

Exercise 2.4: Solution

Consider the situation depicted in Fig. 1 below.

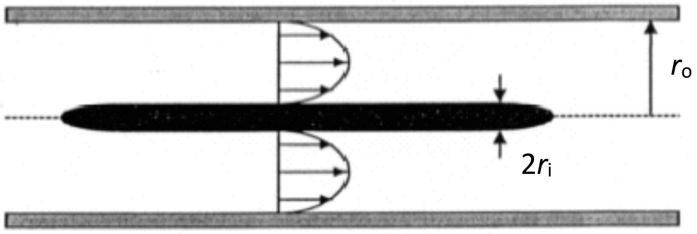

Figure 1. Schema of one of the possible configurations of the improved stent design.

Assuming developed flow, the *x*-momentum equation of the Navier-Stokes equations is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=-\frac{1}{\mu}\frac{\partial p}{\partial x} \tag{1}$$

Integrating twice we obtain

$$u(r)=\frac{1}{4\mu}\frac{\partial p}{\partial x}r^2+c_1\ln r+c_2 \tag{2}$$

Applying the boundary conditions $u(r=r_i)=u(r=r_o)=0$, we come up with the final expression for the velocity distribution:

$$u(r)=\frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2-r_o^2+\frac{r_i^2-r_o^2}{\ln\frac{r_o}{r_i}}\ln\frac{r}{r_o}\right)\quad,\quad r_i\le r\le r_o \tag{3}$$

The flow Q can then be derived by simple integration:

$$Q=\int_{r_i}^{r_o}u(2\pi r)\mathrm{d}r=-\frac{\pi}{8\mu}\frac{\partial p}{\partial x}\left[r_o^4-r_i^4-\frac{\left(r_o^2-r_i^2\right)^2}{\ln\frac{r_o}{r_i}}\right] \tag{4}$$

The shear stress τ acting on the arterial wall is given by

$$\tau=-\mu\left.\frac{\partial u}{\partial r}\right|_{r=r_o} \tag{5}$$

which, using Eq. 3 for $u(r)$, yields

$$\tau = -\frac{1}{4}\frac{\partial p}{\partial x}\left(2r_o + \frac{1}{r_o}\frac{r_i^2 - r_o^2}{\ln\frac{r_o}{r_i}}\right) \tag{6}$$

Equation 6 can also be expressed in terms of the flow Q if we substitute for the pressure gradient from Eq. 4:

$$\tau = -\frac{2\mu}{\pi}\frac{Q}{r_o^4 - r_i^4 - \frac{\left(r_o^2 - r_i^2\right)^2}{\ln\frac{r_o}{r_i}}}\left(2r_o + \frac{1}{r_o}\frac{r_i^2 - r_o^2}{\ln\frac{r_o}{r_i}}\right) \tag{7}$$

To get a "better feeling" of the magnitude of intimal shear stress, it is convenient to normalize it with respect to the shear stress under laminar flow for an open artery and for the same flow Q. This is given by Poiseuille's law:

$$\tau_{\text{Pois}} = -\frac{4\mu Q}{\pi r_o^3} \tag{8}$$

Hence, we can write:

$$\frac{\tau}{\tau_{\text{Pois}}} = \frac{1}{2}\frac{r_o^3}{r_o^4 - r_i^4 - \frac{\left(r_o^2 - r_i^2\right)^2}{\ln\frac{r_o}{r_i}}}\left(2r_o + \frac{1}{r_o}\frac{r_i^2 - r_o^2}{\ln\frac{r_o}{r_i}}\right) \tag{9}$$

If we define γ as

$$\gamma = \frac{r_i}{r_o} \tag{10}$$

Eq. 9 can be rewritten in dimensionless form as

$$\frac{\tau}{\tau_{\text{Pois}}} = \frac{1 + \frac{\gamma^2 - 1}{2\ln\frac{1}{\gamma}}}{1 - \gamma^4 - \frac{\left(1 - \gamma^2\right)^2}{\ln\frac{1}{\gamma}}} \tag{11}$$

The dependence of the relative shear stress τ/τ_{Pois} on parameter γ is shown in Fig. 2 below.

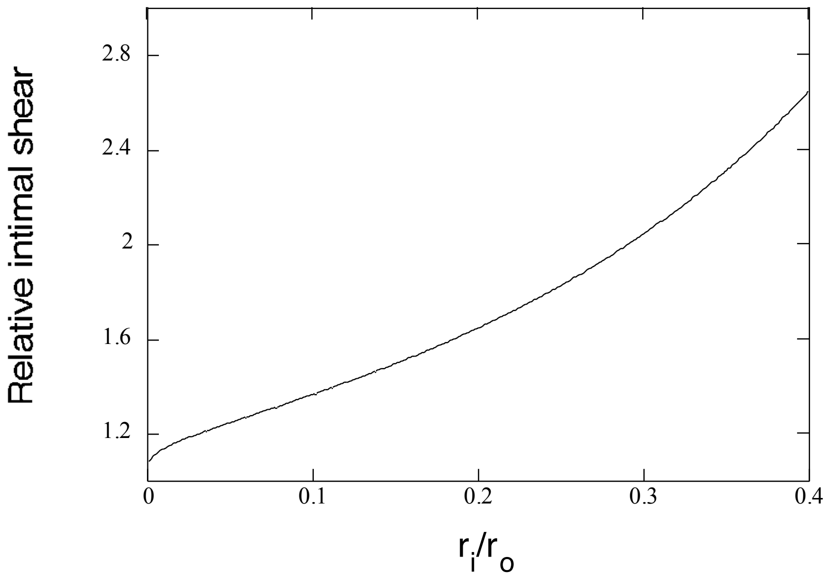

Figure 2. Relative shear stress τ/τ_{Pois} as a function of parameter $\gamma = r_i/r_o$.