Series 2 – Solutions

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Chapter 2: Law of Poiseuille

Exercise 2.1: Solution

From Poiseuille's law derivation we have:

$$\left.\begin{aligned} \tau &= \frac{\Delta p}{\Delta x}\frac{r}{2} \\ \tau &= -b\left(\frac{\partial u}{\partial r}\right)^n \end{aligned}\right\} \Rightarrow \frac{\Delta p}{\Delta x}\frac{r}{2} = -b\left(\frac{\partial u}{\partial r}\right)^n \Rightarrow \frac{\partial u}{\partial r} = \left(-\frac{1}{2b}\frac{\Delta p}{\Delta x}\right)^{\frac{1}{n}} r^{\frac{1}{n}} \Rightarrow$$

$$u = -\frac{n}{n+1}\left(\frac{1}{2b}\frac{\Delta p}{\Delta x}\right)^{\frac{1}{n}} r^{\frac{n+1}{n}} + c \qquad (1)$$

We find the value of constant *c* by applying the boundary condition @ $r = R$:

$$u(r = R) = 0 \Rightarrow c = \frac{n}{n+1}\left(\frac{1}{2b}\frac{\Delta p}{\Delta x}\right)^{\frac{1}{n}} R^{\frac{n+1}{n}}$$

If we substitute the value of *c* into Eq. 1, we obtain the following expression for the velocity:

$$u = \frac{n}{n+1}\left(\frac{1}{2b}\frac{\Delta p}{\Delta x}\right)^{\frac{1}{n}}\left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right)$$

or

$$\frac{u}{u_{max}} = 1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}} \qquad (2)$$

where

$$u_{max} = u(r = 0) = \frac{n}{n+1}\left(\frac{1}{2b}\frac{\Delta p}{\Delta x}\right)^{\frac{1}{n}} R^{\frac{n+1}{n}}$$

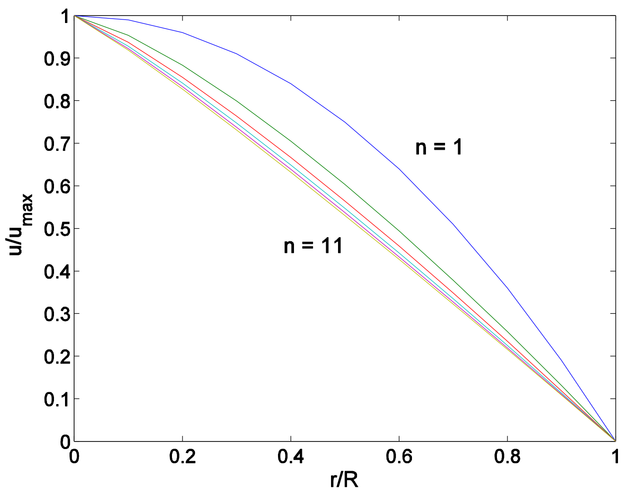
Equation 2 describes qualitatively the velocity distribution along the radius. For $n = 1$, we have parabolic profiles − Poiseuille's type of flow, as can be seen easily from Eq. 2:

$$\frac{u}{u_{max}} = 1 - \left(\frac{r}{R}\right)^{\frac{1+1}{1}} = 1 - \left(\frac{r}{R}\right)^2$$

As *n* increases, the profiles tend to be "more linear". For $n >> 1$, then $n+1 \approx n$ and in that case:

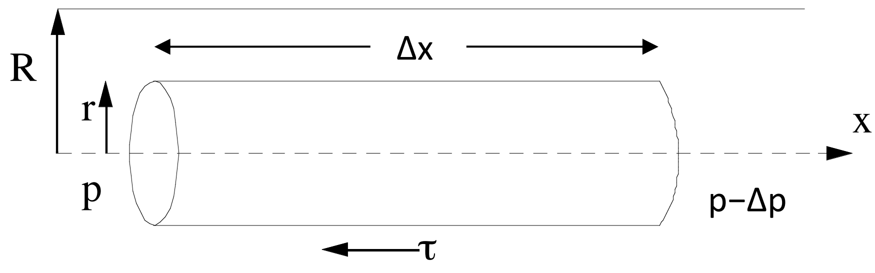
$$\frac{u}{u_{max}} = 1 - \left(\frac{r}{R}\right)^{\frac{n}{n}} = 1 - \frac{r}{R}$$

which is a linear distribution of velocity along *r* (see Figure below).



Exercise 2.2: Solution

a) Let us consider a fluid (any kind of fluid!) flowing in a tube of radius *R*. For generality let us consider only a part of the fluid volume, namely a cylinder of length Δ*x* and radius *r*<*R*, as shown in the figure:



The balance of forces yields

$$p \pi r^2 - (p - \Delta p) \pi r^2 = 2 \pi r \Delta x \tau$$

or

$$\tau = \frac{\Delta p}{\Delta x}\frac{r}{2} \qquad (1)$$

which is valid for all *r*. Thus, for a given pressure drop (Δ*p*/Δ*x*) the distribution of shear stress is given and is independent of the fluid properties.

b) The equation relating the shear stress to shear rate for blood is given in the course notes:

$$\sqrt{\tau} = \sqrt{\tau_y} + \sqrt{\mu\gamma}$$

Solving for the shear rate we obtain

$$\gamma = \frac{\left(\sqrt{\tau} - \sqrt{\tau_y}\right)^2}{\mu} = \frac{\left(\sqrt{\frac{r}{2}\frac{\Delta p}{\Delta x}} - \sqrt{\tau_y}\right)^2}{\mu} \quad , \quad \text{with} \quad \gamma = -\frac{\partial u}{\partial r}$$

Note that the pressure gradient ($\Delta p/\Delta x$) is a constant, independent of x and r. Thus, the above differential equation can be integrated with respect to r to yield:

$$\frac{\partial u}{\partial r} = -\frac{1}{\mu}\left(\frac{\Delta p}{\Delta x}\frac{r}{2} + \tau_y - \sqrt{2\tau_y \frac{\Delta p}{\Delta x}}\sqrt{r}\right) \Rightarrow$$

$$u = -\frac{1}{\mu}\left(\frac{\Delta p}{\Delta x}\frac{r^2}{4} + \tau_y r - \frac{2}{3}\sqrt{2\tau_y \frac{\Delta p}{\Delta x}} r^{\frac{3}{2}}\right) + c \qquad (2)$$

We find the value of constant c by applying the boundary condition @ $r = R$:

$$u(r = R) = 0 \Rightarrow c = \frac{1}{\mu}\left(\frac{\Delta p}{\Delta x}\frac{R^2}{4} + \tau_y R - \frac{2}{3}\sqrt{2\tau_y \frac{\Delta p}{\Delta x}} R^{\frac{3}{2}}\right)$$

If we substitute the value of c into Eq. 2, we obtain the following expression for the velocity:

$$u = \frac{1}{\mu}\left[\frac{\Delta p}{\Delta x}\frac{R^2 - r^2}{4} + \tau_y (R - r) - \frac{2}{3}\sqrt{2\tau_y \frac{\Delta p}{\Delta x}}\left(R^{\frac{3}{2}} - r^{\frac{3}{2}}\right)\right] \qquad (3)$$

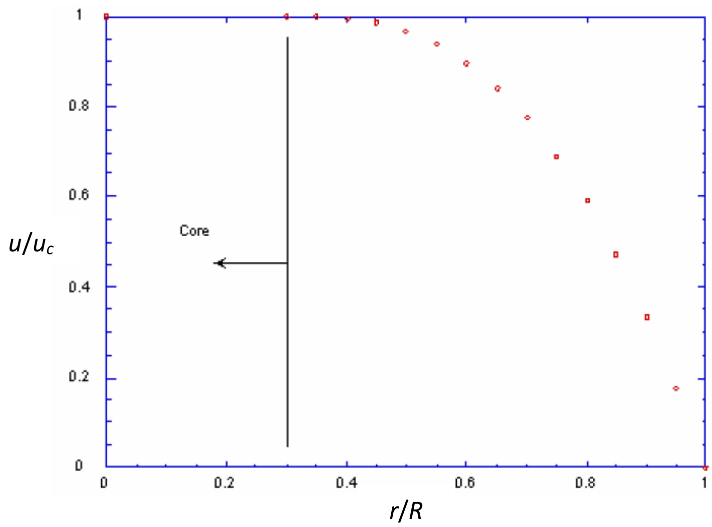
The above profile would have been equal to Poiseuille (parabolic) profile if the yield stress, τ_y, was equal to zero (verify). Now a certain yield stress exists, which means that below the yield stress level the fluid will behave as a solid. Since shear stress is highest at the wall and smallest towards the middle, there will be a part of the fluid, the core, occupying the central part of the tube up to a radius, r_c, where the fluid will be moving as a rigid body. The radius r_c is the point where shear stress equals the yield stress, therefore, by virtue of Eq. 1:

$$\tau_y = \frac{\Delta p}{\Delta x}\frac{r_c}{2} \Rightarrow r_c = 2\,\tau_y \left(\frac{\Delta p}{\Delta x}\right)^{-1} \qquad (4)$$

As the pressure gradient increases, more of the fluid in the tube shears and thus the extent of core diminishes. Using Eqs 3 and 4, the velocity can be expressed by means of r_c:

$$u(r) = \frac{1}{4\mu}\frac{\Delta p}{\Delta x}\left[R^2 - r^2 + 2r_c (R - r) - \frac{8}{3}\sqrt{r_c}\left(R^{\frac{3}{2}} - r^{\frac{3}{2}}\right)\right]$$

A typical profile for a fluid with $r_c / R = 0.3$ is given in the figure below:



c) The core velocity is the velocity at $r = r_c$:

$$u_c = \frac{1}{4\mu}\frac{\Delta p}{\Delta x}\left[R^2 + 2r_c R - \frac{1}{3}r_c^2 - \frac{8}{3}\sqrt{r_c R^3}\right]$$

d) Flow is obtained by integrating the velocity over the cross-sectional area of the tube

$$Q = 2\pi\int_0^R ur\mathrm{d}r = \pi r_c^2 u_c + 2\pi\int_{r_c}^R ur\mathrm{d}r$$

which finally yields:

$$Q = \frac{\pi R^4}{8\mu}\frac{\Delta p}{\Delta x}\left[1 - \frac{16}{7}\left(\frac{2\tau_y}{R\frac{\Delta p}{\Delta x}}\right)^{\frac{1}{2}} + \frac{4}{3}\left(\frac{2\tau_y}{R\frac{\Delta p}{\Delta x}}\right) - \frac{1}{21}\left(\frac{2\tau_y}{R\frac{\Delta p}{\Delta x}}\right)^4\right]$$

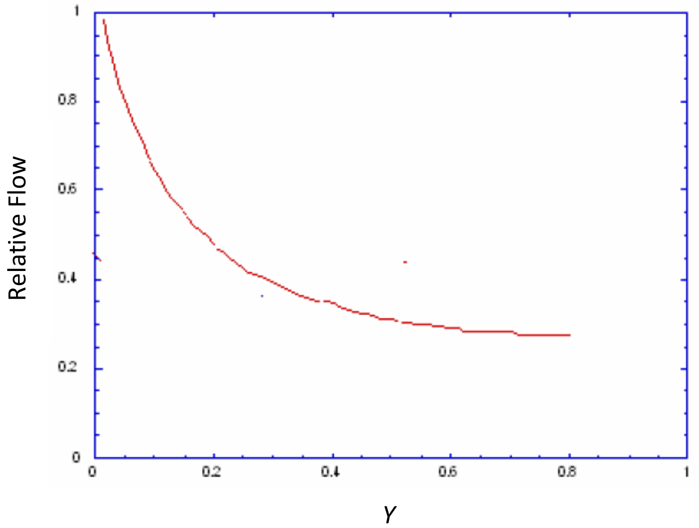
The terms outside the brackets give exactly the relation for Poiseuille flow. The terms inside the brackets, therefore, represent the change (diminish) in flow, as compared to Poiseuille flow. Defining Y as the relative strength of the yield stress

$$Y = \frac{2\tau_y}{R\frac{\Delta p}{\Delta x}}$$

and plotting the relative flow

$$\frac{Q}{\frac{\pi R^4}{8\mu}\frac{\Delta p}{\Delta x}}$$

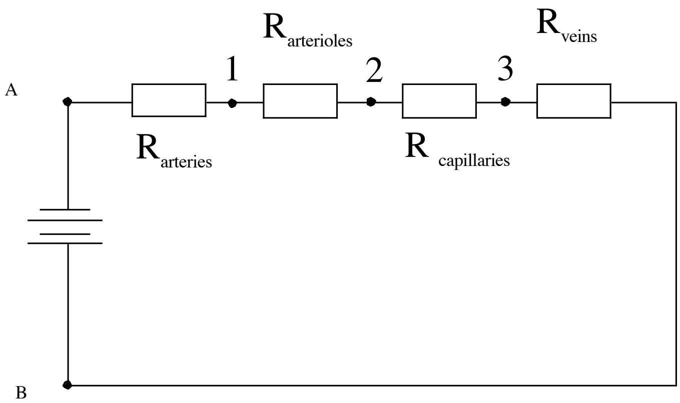
as a function of *Y*, we obtain:



Chapter 6: Resistance

Exercise 6.1: Solution

In this problem, the systemic circulation is represented by a series of resistors. Voltage is analogous to pressure, current to flow and electrical resistance to fluid resistance.



The voltage source is the heart, with the “voltage” at point A being the mean arterial pressure, or 100 mmHg. The “voltage” at point B is 6 mmHg. So, the “voltages” at points 1, 2 and 3 are:

$$V_1 = V_A - \frac{(V_A - V_B)\,R_{arteries}}{R_{total}} = 100 - (100-6) \cdot 0{,}19 = 82 \text{ mmHg}$$

$$V_2 = V_1 - \frac{(V_A - V_B) R_{arterioles}}{R_{total}}$$ = 82 - (100-6) · 0,47 = 38 mmHg

$$V_3 = V_2 - \frac{(V_A - V_B) R_{capillaries}}{R_{total}}$$ = 38 - (100-6) · 0,27 = 13 mmHg

where $R_{total} = R_{arteries} + R_{arterioles} + R_{capillaries} + R_{veins}$.

Graphically, the pressure distribution is: