Series 1 – Solutions

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Chapter 1.3: Introduction to the CV system

Exercise 1.3.1 - Solution

a) Assume:

- Flow conditions for ascending aorta: mean cardiac output = $5.5\ \mathrm{lt/min}$.
- Flow waveform is a constant $3 \cdot 5.5\ \mathrm{lt/min}$ for 1/3 of the cardiac cycle and zero for the rest.

$$V_{aort} = \frac{Q_{aort}}{A_{aort}} = \frac{3 \cdot 5.5 \frac{\mathrm{lt}}{\mathrm{min}}}{5.3\ \mathrm{cm}^2} \cdot \frac{1000\ \frac{\mathrm{cm}^3}{\mathrm{lt}}}{60 \frac{\mathrm{sec}}{\mathrm{min}}} = 52 \frac{\mathrm{cm}}{\mathrm{sec}}$$

This is the mean systolic, spatially averaged velocity.

b) Assume: Flow = $5.5\ \mathrm{lt/min}$, constant in capillaries.

$$V_{cap} = \frac{Q_{cap}}{A_{cap}} = \frac{5.5 \frac{\mathrm{lt}}{\mathrm{min}}}{3500\ \mathrm{cm}^2} \cdot \frac{1000\ \frac{\mathrm{cm}^3}{\mathrm{lt}}}{60 \frac{\mathrm{sec}}{\mathrm{min}}} = 0.026 \frac{\mathrm{cm}}{\mathrm{sec}}$$

b) Assume: Flow = $5.5\ \mathrm{lt/min}$, constant in venae cavae.

$$V_{vc} = \frac{Q_{vc}}{A_{vc}} = \frac{5.5 \frac{\mathrm{lt}}{\mathrm{min}}}{18\ \mathrm{cm}^2} \cdot \frac{1000\ \frac{\mathrm{cm}^3}{\mathrm{lt}}}{60 \frac{\mathrm{sec}}{\mathrm{min}}} = 5.1 \frac{\mathrm{cm}}{\mathrm{sec}}$$

Exercise 1.3.2 - Solution

Based on the course figure entitled «Characteristics of Circulatory Vessels», the total volume of the systemic circulation (without heart) is about 4,4 lt. Also, the cardiac output is about $5.5\ \mathrm{lt/min}$. So, a blood particle traverses the whole arterial and venous circulation in about:

$$\frac{4.4\ \mathrm{lt}}{5.5 \frac{\mathrm{lt}}{\mathrm{min}}} \cdot 60 \frac{\mathrm{sec}}{\mathrm{min}} = 48\ \mathrm{sec}$$

Exercise 1.3.3 - Solution

See course slide «standing adult with raised hand». The mean arterial pressure in the raised hand is about 40 mmHg. The mean arterial pressure in the foot is about 170 mmHg. The mean venous pressure in the raised hand is about -50 mmHg. The mean venous pressure in the foot is about 80 mmHg.

The negative venous pressure (-50 mmHg) in the hand prevents the blood from returning to the heart.

Notes:

a) The presence of valves in the venous system, the contractions of the vascular smooth muscle cells, the compression exerted by the skeletal muscles, and the respiration assure the venous return.

b) This kind of static estimation does not take into account the pressure modifications due to viscous and inertial losses during the wave propagation in the arterial system.

c) Because of the low pressure dominating in the venous system, one can think that the veins collapse and block the return of blood. In fact, they take the shape of « 8 » leaving a small passage for the return of blood. Furthermore, the veins are supported by the surrounding tissues.

Chapter 1.4: Blood Rheology

Exercise 1.4.1 – Solution

(a) The Navier-Stokes equations in cylindrical coordinates are as follows:

r-direction

$$\rho\left(\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}-\frac{{V_\theta}^2}{r}+V_z\frac{\partial V_r}{\partial z}\right)=\rho g_r-\frac{\partial p}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2}+\frac{\partial^2 V_r}{\partial z^2}-\frac{V_r}{r^2}-\frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}\right)$$

θ-direction

$$\rho\left(\frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_\theta V_r}{r}+V_z\frac{\partial V_\theta}{\partial z}\right)=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rV_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2}+\frac{\partial^2 V_\theta}{\partial z^2}-\frac{2}{r^2}\frac{\partial V_r}{\partial \theta}\right)$$

z-direction

$$\rho\left(\frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}+V_z\frac{\partial V_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2}+\frac{\partial^2 V_z}{\partial z^2}\right)$$

Assume:

- Steady flow: $\frac{\partial}{\partial t}=0$
- Velocity components V_r and V_z are zero (fluid moves in a circular pattern)
- Velocity component V_θ depends only on r (this derives from the continuity equation).
- Gravitational terms can be neglected.

After cancelling out the zero terms from the Navier-Stokes equations we are left with:

r-direction

$$\rho \frac{V_\theta^2}{r} = \frac{\partial p}{\partial r}$$

θ-direction

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(rV_\theta)}{dr}\right] = 0 \qquad (1)$$

z-direction

$$\frac{\partial p}{\partial z} = 0$$

Integration of Eq. (1) with respect to r yields:

$$V_\theta(r) = c_1\frac{r}{2} + c_2\frac{1}{r} \qquad (2)$$

Where c_1 and c_2 are constants. Then applying the boundary conditions:

$$V_\theta(r = R_e) = \Omega R_e, \qquad V_\theta(r = R_i) = 0 \qquad (3)$$

to Eq. (2), we obtain the values of the constants:

$$c_1 = \frac{2\Omega {R_e}^2}{{R_e}^2 - {R_i}^2}, \quad c_2 = -\frac{\Omega {R_e}^2 {R_i}^2}{{R_e}^2 - {R_i}^2} \qquad (4)$$

Substitution of (4) into (2) gives us:

$$V_\theta(r) = \frac{2\Omega {R_e}^2}{{R_e}^2 - {R_i}^2}\frac{r}{2} - \frac{\Omega {R_e}^2 {R_i}^2}{{R_e}^2 - {R_i}^2}\frac{1}{r} \qquad (5)$$

Substitution of (5) into the expression for the shear stress

$$\tau = \mu\gamma = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]_{r=R_i} \qquad (6)$$

gives us the shear stress acting on the surface of the inner cylinder:

$$\tau = \mu\frac{2\Omega {R_e}^2}{{R_e}^2 - {R_i}^2} \qquad (7)$$

The torque, M, applied on the inner cylinder is calculated as he product of the applied force times the lever arm:

$$M = (2\pi R_i h)\cdot\tau\cdot R_i \qquad (8)$$

If we substitute (7) into (8), we obtain the expression relating viscosity, torque and rotational speed:

$$\mu = \frac{M(R_e^2 - R_i^2)}{4\pi\Omega h R_i^2 R_e^2}$$

(**b**) For a linear velocity profile:

$$\tau = \mu \frac{\partial v_\theta}{\partial r} \cong \mu \frac{V_\theta(r = R_e)}{R_e - R_i} = \mu \frac{\Omega R_e}{R_e - R_i} \quad (9)$$

If we take Eq. (7):

$$\tau = \mu \frac{2\Omega {R_e}^2}{{R_e}^2 - {R_i}^2} = \mu \frac{2\Omega {R_e}^2}{(R_e - R_i)(R_e + R_i)} \quad (10)$$

If we assume a small gap $(R_e + R_i) \cong 2R_e$ and Eq. (10) becomes equal to Eq. (9). Therefore, the approximation of a linear velocity profile is good, if the gap $\Delta R = R_e - R_i$ is small compared to R_e.