Series 10 – Solutions

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Chapter 20: Wave Travel and Reflection

Exercise 21.1 – Solution

1) The reflected power P_r at a reflection site is proportional to the square of the reflection coefficient R. For two tubes of slightly different inner diameters ($D_1 \approx D_2$ with $D_1 > D_2$) the reflection coefficient will be:

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{1}$$

where $Z_n = \rho \cdot c/A_n$ (n = 1,2) is the characteristic impedance (blood density ρ times pulse wave velocity c divided by the lumen area $A_n = \pi D_n^2/4$). If the lumen areas A_1 and A_2 are nearly equal (as are the densities and wave speeds) the impedances Z_1 and Z_2 will be nearly equal. This gives a reflection coefficient which is very small ($R << 1$). Since the square of a small number is even smaller, the reflected power or energy (integral of power) will be nearly zero.

2) In order to show that the pressure p_n depends on the inverse of the diameter D_n, one must consider from question 1 that almost all of the power is transferred across each reflection site. The power is:

$$P = \frac{p_n^2}{Z_n} \tag{2}$$

which is equal to a constant $P = ct$ if all of the power is transmitted along the tube (no power is reflected). Therefore,

$$\frac{p_n^2}{Z_n} = ct \Rightarrow p_n = ct\sqrt{Z_n} \propto \sqrt{\frac{\rho c}{A_n}} \propto \frac{1}{D_n} \tag{3}$$

Hence, as the diameter D_n decreases the pressure amplitude p_n increases.

3) For the flow Q_n:

$$Q_n = \frac{p_n}{Z_n} = \frac{ct\sqrt{Z_n}}{Z_n} = \frac{ct}{\sqrt{Z_n}} = \frac{ct}{\sqrt{\frac{\rho c}{A_n}}} \propto D_n \tag{4}$$

But, the average velocity $\overline{U}_n$ would be:

$$\overline{U}_n = \frac{Q_n}{A_n} = \frac{Q_n}{\frac{\pi}{4} D_n^2} \propto \frac{D_n}{D_n^2} \propto \frac{1}{D_n} \tag{5}$$

Hence, as the diameter D_n decreases the average velocity $\overline{U}_n$ increases.

Exercise 21.2 – Solution

1. In the first part of this problem, the objective is to calculate the pressure at any point *z* due to multiple reflections of an initial wave. The primary, non-attenuated wave is:

$$p_1 = p_0\, \mathrm{e}^{\mathrm{i}\omega t - \gamma z} \tag{1}$$

where γ=ib. The first reflected wave travels back from *B* to *A*:

$$p_2 = R_2 p_0\, \mathrm{e}^{\mathrm{i}(\omega t - 2bL + bz)} \tag{2}$$

The second reflection occurs at point *A*, going back toward *B*:

$$p_3 = R_1 R_2 p_0\, \mathrm{e}^{\mathrm{i}(\omega t - 2bL - bz)} \tag{3}$$

The third:

$$p_4 = R_1 R_2^2 p_0\, \mathrm{e}^{\mathrm{i}(\omega t - 4bL + bz)} \tag{4}$$

and so on. The total pressure at any point *z* will be the sum of all the above waves, or

$$p(z,t) = p_1 + p_2 + p_3 + \ldots \tag{5}$$

After adding the waves together, the sum may be separated into forward and backward traveling parts:

$$p(z,t) = p_0\, \mathrm{e}^{\mathrm{i}(\omega t - bz)} \left(1 + R_1 R_2\, \mathrm{e}^{-2\mathrm{i}bL} + \ldots\right) + R_2 p_0\, \mathrm{e}^{\mathrm{i}(\omega t - 2bL + bz)} \left(1 + R_1 R_2\, \mathrm{e}^{-2\mathrm{i}bL} + \ldots\right) \tag{6}$$

Using the relationship

$$1 + a + a^2 + a^3 + \ldots = 1/(1-a) \tag{7}$$

the terms in the brackets may be simplified:

$$p(z,t) = p_0 \frac{e^{\mathrm{i}(\omega t - bz)} + R_2 e^{\mathrm{i}(\omega t - 2bL + bz)}}{1 - R_1 R_2 e^{-2\mathrm{i}bL}} \tag{8}$$

2. If the reflection coefficients are $R_1 = R_2 = 1$, the Eq. (8) may be simplified. Multiplying by e^{ibL} and using the complex exponential definition $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, Eq. (8) will give:

$$p(z,t) = p_0 e^{i\omega t} \frac{\cos\left[b(L-z)\right]}{i\sin(bL)} \tag{9}$$

This is referred to as "standing wave" because it does not propagate along the artery. The pressure oscillates at each point along the artery, but the spatial waveform remains the same. Note that if $bL = n\pi$, where n is an integer, the amplitude will become very large (resonance).

3. For the case of an attenuated wave, one may simply substitute a complex expression for the wave number γ:

$$\gamma = a + ib \tag{10}$$

The pressure at any point, from Eq. 8, is:

$$p(z,t) = p_0 \frac{e^{i(\omega t - bz) - az} + R_2 e^{i(\omega t - 2bL + bz) + az}}{1 - R_1 R_2 e^{-2ibL - 2aL}} \tag{11}$$