ME-474 Numerical Flow Simulation

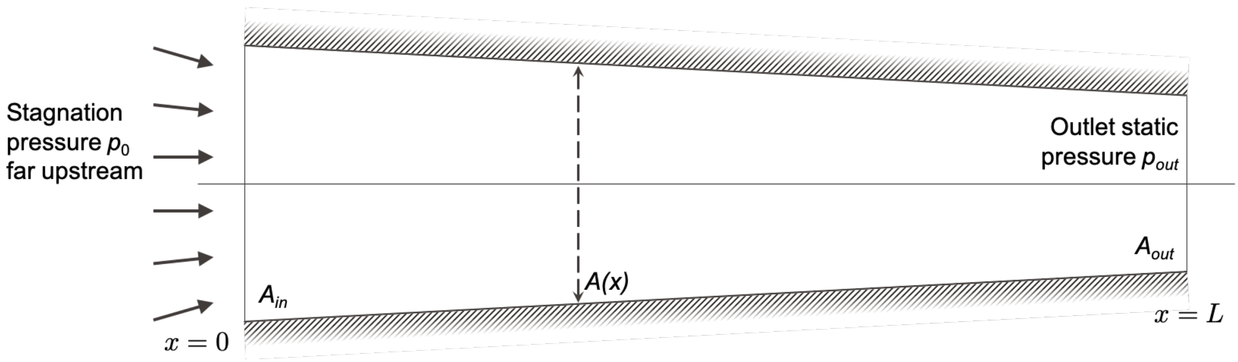
Assessment #1

Fall 2024

Consider the steady incompressible flow through a 2D planar, converging nozzle. The flow is assumed to be frictionless (viscosity is neglected) and unidirectional along the x direction, with all flow variables uniformly distributed throughout every cross-section. The problem can be reduced to 1D, and the flow is governed by the 1D steady inviscid Navier-Stokes equations:

$$\text{Continuity eq. (mass conservation):} \quad \frac{d(\rho u)}{dx} = 0,$$

$$\text{Momentum eq. (momentum conservation):} \quad \frac{d(\rho u u)}{dx} = -\frac{dp}{dx}.$$

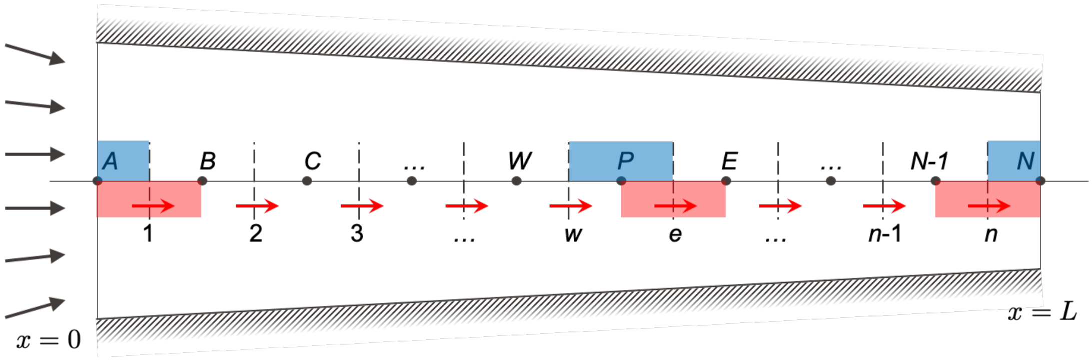


Implement a Matlab code to solve this problem numerically in terms of pressure and velocity, using the finite volume method, a staggered grid arrangement, the upwind differencing scheme (UD) for the momentum convective term, and the SIMPLE algorithm for pressure-velocity coupling. Present your results by comparing the numerical solution (pressure, velocity, and mass flow rate) with the analytical solution of the problem (see solution of exercise 5).

Data and guidelines:

- Length $L = 2$ m.
- Constant density $\rho = 1$ kg/m^3.
- Inlet cross-section area $A_{in} = 0.5$ m^2. Outlet cross-section area $A_{out} = 0.1$ m^2. The area $A(x)$ varies linearly with distance from the nozzle inlet.
- Inlet boundary condition: assume that the flow entering the nozzle is drawn from a large chamber. Far away upstream from the nozzle inlet, the fluid has zero momentum (fluid at rest) and the stagnation pressure is $p_0 = 10$ Pa.
- Outlet boundary condition: static pressure $p_{out} = 0$ Pa.

  (Note that pressure is relative, because only its gradient appears in the equation. In other words, only the overall pressure difference $p_0 - p_{out}$ matters. One can add a constant to p_0 and p_{out}, and would obtain the same velocity field $u(x)$ and pressure gradient dp/dx.)



- Discretize the domain $x \in [0, L]$ with N equidistant pressure nodes (original grid, N control volumes) and $n = N - 1$ equidistant velocity nodes (staggered grid, $n = N - 1$ inner faces). Define the two grids as:

$$x_p = 0,\ \Delta x,\ 2\Delta x, \dots, (N-1)\Delta x = L,$$

$$x_u = \Delta x/2,\ 3\Delta x/2, \dots, L - \Delta x/2.$$

- It may be convenient to define two vectors of cross-section areas at the node locations and face locations. Given the expression of $A(x)$ (which you have to find), you can build the two vectors $A_p = A(x_p)$ and $A_u = A(x_u)$.

- Initialize the velocity and pressure fields. For velocity, take a guess flow rate (e.g. $\dot{m} = 1$ kg/s) and compute

$$u^{old}(x_u) = \frac{\dot{m}}{\rho A_u}.$$

  For pressure, we want an initial guess that already satisfies the outlet boundary condition so that we will not need a pressure correction at the outlet, i.e. $p'_N = 0$. Assume for instance a linearly decreasing pressure:

$$p^*(x_p) = p_0 - (p_0 - p_{out})\frac{x_p}{L}.$$

- Solve the linear system $\mathbf{M}_u \mathbf{u}^* = \mathbf{b}_u$ associated with the discretized momentum equation:

  – **Inner velocity CVs** 2 to $n - 1$:

$$\boxed{a_e u_e^* = a_w u_w^* + S}$$

  where

$$M_u(i,i) = a_e = F_E = \rho\frac{u_e^{old} + u_{ee}^{old}}{2}A_E = \rho\frac{u^{old}(i) + u^{old}(i+1)}{2}A_p(i+1),$$

$$M_u(i,i-1) = -a_w = -F_P = -\rho\frac{u_w^{old} + u_e^{old}}{2}A_P = -\rho\frac{u^{old}(i-1) + u^{old}(i)}{2}A_p(i),$$

$$b_u(i) = S = (p_P^* - p_E^*)A_e = (p^*(i) - p^*(i+1))A_u(i).$$

  – **Inlet velocity CV 1:**

$$\boxed{a_1 u_1^* = S_u}$$

  where

$$M_u(1,1) = a_1 = F_B + \frac{1}{2}F_A\left(\frac{A_1}{A_A}\right)^2, \quad A_1 = A_u(1), \quad A_A = A_p(1),$$

$$F_B = \rho\frac{u^{old}(1) + u^{old}(2)}{2}A_p(2), \quad F_A = \rho u^{old}(1)A_u(1),$$

$$b_u(1) = S_u = (p_0 - p_B^*)\,A_1 + F_A\frac{A_1}{A_A}u_1^{old} = (p_0 - p^*(2))\,A_1 + F_A\frac{A_1}{A_A}u^{old}(1).$$

- **Outlet velocity CV** n**:**

$$\boxed{a_n u_n^* = a_{n-1} u_{n-1}^* + S}$$

  where

$$M_u(n,n) = a_n = F_N = \rho u_n^{old} A_n = \rho u^{old}(n) A_u(n),$$

$$M_u(n,n-1) = -a_{n-1} = -F_{N-1} = -\rho \frac{u_{n-1}^{old} + u_n^{old}}{2} A_{N-1} = -\rho \frac{u^{old}(n-1) + u^{old}(n)}{2} A_p(N-1),$$

$$b_u(n) = S = (p_{N-1}^* - p_N^*) A_n = (p^*(N-1) - p^*(N)) A_u(n).$$

- Solve the linear system $\mathbf{M}_p p' = \mathbf{b}_p$ associated with the discretized pressure correction equation:
  - **Inner pressure CVs** 2 to $N-1$:

$$\boxed{a_P p_p' = a_W p_W' + a_E p_E' + b}$$

  where

$$M_p(i,i) = a_W + a_E,$$

$$M_p(i,i-1) = -a_W = -\rho d_w A_w = -\rho d_u(i-1) A_u(i-1),$$

$$M_p(i,i+1) = -a_E = -\rho d_e A_e = -\rho d_u(i) A_u(i),$$

  and

$$d_u(i) = \frac{A_u(i)}{a_e} \quad \text{for } i = 2 \dots n, \quad d_u(1) = \frac{A_u(1)}{a_1}, \quad d_u(n) = \frac{A_u(n)}{a_n},$$

$$b_p(i) = b = \rho u_w^* A_w - \rho u_e^* A_e = \rho A_u(i-1) u^*(i-1) - \rho A_u(i) u^*(i).$$

  - **Inlet/outlet pressure CVs** 1 and N:

$$\boxed{p'(1) = p'(N) = 0,}$$

  so

$$M_p(1,1) = M_p(N,N) = 1, \quad b_p(1) = b_p(N) = 0.$$

- Correct pressure and velocity:

$$u^{calc}(i) = u^*(i) + d_u(i)(p'(i) - p'(i+1)),$$

$$p^{calc}(i) = p^*(i) + p'(i) \quad \text{for } i \geq 2,$$

$$p^{calc}(1) = p_0 - \frac{1}{2}\rho \left(u^{calc}(1) \frac{A_u(1)}{A_p(1)} \right)^2.$$

- Calculate the relative residual of the momentum equation, and the absolute RHS of the pressure correction equation:

$$r_u = \frac{||\mathbf{M_u} \mathbf{u}^{calc} - \mathbf{b}_u||}{||diag(\mathbf{M_u}) \cdot \mathbf{u}^{calc}||}, \quad r_p = ||\mathbf{b}_p||.$$

- Update with under-relaxation:

$$\mathbf{u}^{new} = \alpha_u \mathbf{u}^{calc} + (1-\alpha_u) \mathbf{u}^{old},$$

$$p^{new} = \alpha_p p^{calc} + (1-\alpha_p) p^*.$$

- Iterate. At each new iteration, use the solution from the end of the previous iteration as new guess solution: the previous $\mathbf{u}^{new}$ becomes the new guess velocity $\mathbf{u}^{old}$, and the previous p^{new} becomes the new guess pressure p^*.

- Stop at convergence, i.e. when both r_u and r_p are smaller than a given tolerance. For the tolerance, take for instance the threshold value 10^{-6}.

- Evaluate the solution accuracy:
  - Check the final values of r_u and r_p.
  - Check the final global mass balance: the ratio

$$\frac{\dot{m}_{in} - \dot{m}_{out}}{\text{mean}(\dot{m})}$$

    should be small.
  - Compare the final numerical $u(x)$ and $p(x)$ to the analytical solution. Compute the relative error.

- Write a short report (a single pdf file) presenting your results:
  - Plots of final numerical and analytical $u(x)$ and $p(x)$,
  - Plot of r_u and r_p against iteration number,
  - Value of final global mass balance,
  - Values of relative errors of the final numerical velocity, pressure, and flow rate compared to the analytical solution.

  Report your results for $N = 21$ nodes and for under-relaxation coefficients $\alpha_u = \alpha_p = 0.1$. If you wish (not mandatory), you can try additional values of N, α_u, α_p to see how they affect the convergence rate, stability, and accuracy.

  Provide some concise explanations and comments. Numerical values and figures without comments are virtually useless.

  Make sure the figures are readable and clear, which means you must put labels, choose suitable ranges, scales and fonts, etc.