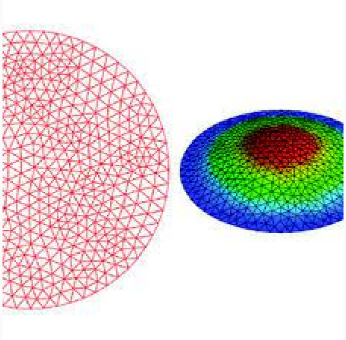
Dynamic analysis of Kirchhoff plates

Classical structural elements



ME473 Dynamic finite element analysis of structures

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2025

Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation
3		FEM global	Project 1 statement
4		FEM local	
5		FEM local	Project 1 submission
6	Classical structural elements	Bars and trusses	Project 2 statement
7		Beams	
8		Frames and grids	
9		Kirchhoff-Love plates	Project 2 submission

Summary

- Kirchhoff-Love plate theory
- Shell elements
- Example: first fundamental frequency of simply supported plate

Recommended readings

(L) Logan, A first course in the finite element method, 6th ed. (chap. 12)

(N) Neto et al., Engineering Computation of Structures (chap. 6)

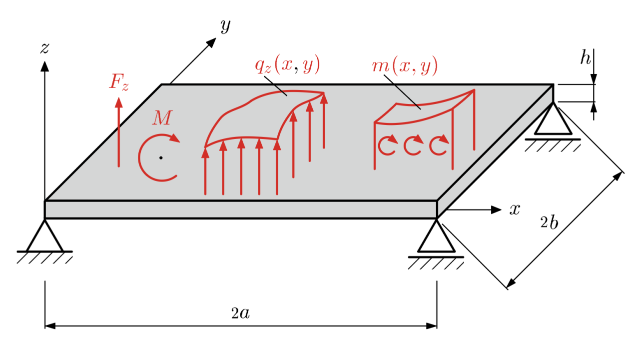
(O) Ochsner, PDE for classical structural members (chap. 6)

(G) Gmür, Dynamique des structures (chap. 3)

Classic plate theory

Plate structure

- Plate structures are geometrically similar to structures of the 2D plane stress problem, but it usually carries only transversal loads that lead to bending deformation of the plate.
- For example: floors of a building, aerospace and ships structures, etc...



(Credit: (O))

Plate models

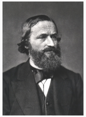
Kirchhoff (1888) and Love (1945)

- *Shear free plates*: thin plates where the contribution of shear force on the deformations is neglected.
- Two-dimensional extension of the Bernoulli-Euler beam theory.

Mindlin (1951) and Reissner (1945)

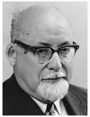
- *Shear deformable plates*: thick plates where the contribution of shear force on deformations is considered.
- Two-dimensional extension of the Timoshenko beam theory.

Gustav Kirchhoff



1888

Augustus Love
Eric Reissner



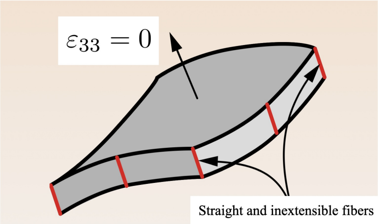
1945

Raymond Mindlin

1951

Geometry assumptions

- The thickness of the plate h is constant and much smaller than the planar dimensions a and b: $h/a < 0.1$ and $h/b < 0.1$.
- **Inextensibility of transverse fibers**: h is constant and $\varepsilon_{33} = 0$,



(Credit:(G))

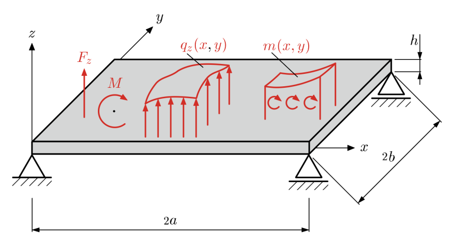
Material and loads assumptions

Material

- The material is homogenous and linear-elastic according to Hooke's law for a plane stress state ($\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$),

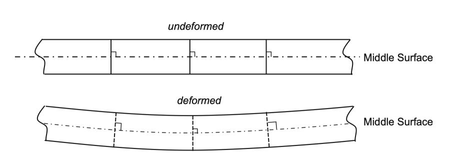
Loads

- External forces act only perpendicular to the $x - y$ plane, the vector of external moments lies within the $x - y$ plane.
- Displacement $u_3(x, y, t)$ is small compared to h: $u_3 < 0.2h$.



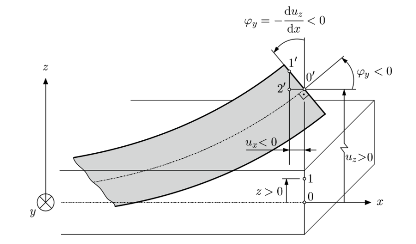
Kirchhoff assumption

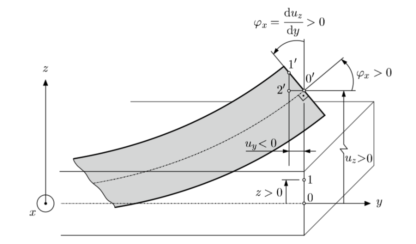
Rectilinearity of the normals: Bernoulli's hypothesis is valid, i.e. a cross-sectional plane stays plane and unwrapped in the deformed state.
This means that the shear strains ε_{13} and ε_{23} due to the distributed shear forces q_x and q_y are neglected.



A straight fiber that is perpendicular to the middle plane of the plate before deformation remain straight and normal to it after deformation.

(Credit: (N))





$$-\varphi_2 \approx \sin(-\varphi_2) = -\frac{u_1}{z}$$

$$-\varphi_2 \approx \tan(-\varphi_2) = \frac{du_3}{dx}$$

$$\implies u_1 = -z\frac{du_3}{dx}$$

$$\varphi_1 \approx \sin(\varphi_1) = -\frac{u_2}{z}$$

$$\varphi_1 \approx \tan(\varphi_1) = \frac{du_3}{dy}$$

$$\implies u_2 = -z\frac{du_3}{dx}$$

Transverse displacement u_3 is the only independent variable:

$$\mathbf{u} = \begin{bmatrix} -z\frac{\partial u_3}{\partial x} \\ -z\frac{\partial u_3}{\partial y} \\ u_3 \end{bmatrix}$$

Deformation is exaggerated in the figures for better illustration.

Strain-displacement relation

- Using classical engineering definitions of strain:

$$\varepsilon_{ii} = \partial_i u_i \qquad \text{and} \qquad \gamma_{ij} = \partial_i u_j + \partial_j u_i$$

we obtain

$$\underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}}_{\boldsymbol{\varepsilon}} = -z \underbrace{\begin{bmatrix} \partial^2_{xx} \\ \partial^2_{yy} \\ 2\partial^2_{xy} \end{bmatrix}}_{\boldsymbol{\nabla}_k} u_3 = z \underbrace{\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}}_{\boldsymbol{\kappa}}$$

- $\boldsymbol{\kappa}$ is the matrix that contains the changes in the curvature of the plate, given as $\boldsymbol{\kappa} = -\boldsymbol{\nabla}_k u_3$.
- Note that $\varepsilon_{12} = \varepsilon_{23} = 0$ due to Kirchhoff assumptions and $\varepsilon_{33} = 0$ due to the inextensibility of transverse fibers assumption.

Constitutive equation for isotropic material

- Classical plate theory assumes a plane stress state: $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$.
- Constitutive equation for isotropic material is $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$ or $\boldsymbol{\varepsilon} = \mathbf{D}\boldsymbol{\sigma}$ where

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix},$$

or

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(\nu+1) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}.$$

- $\mathbf{C}$ is the *elasticity matrix* and $\mathbf{D} = \mathbf{C}^{-1}$ is the *elastic compliance matrix*.

Constitutive equation for orthotropic material

- The constitutive equation for orthotropic material is $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$ or $\boldsymbol{\varepsilon} = \mathbf{D}\boldsymbol{\sigma}$ where

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

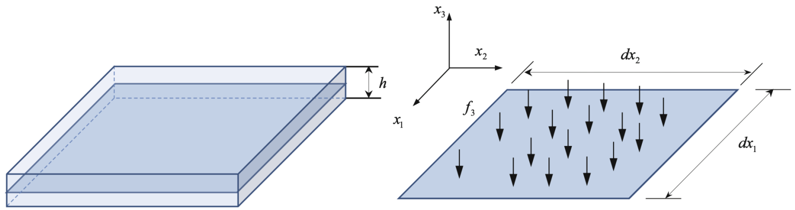
  where

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{33} = G_{12}$$

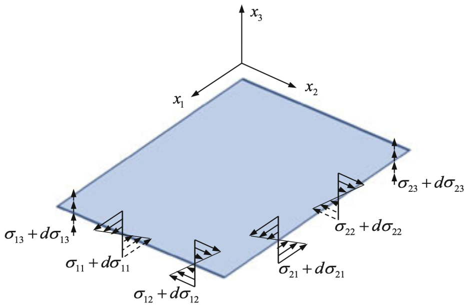
- The orthotropic properties of the lamina are given: E_1, E_2, ν_{12}, G_{12} and $\nu_{21} = \nu_{12}E_2/E_1$ applies.

External forces

Consider a plate cell of dimensions $dx_1 \times dx_2 \times h$ that is submitted to external forces, here denoted by f_3, and inertial force proportional to the material density.



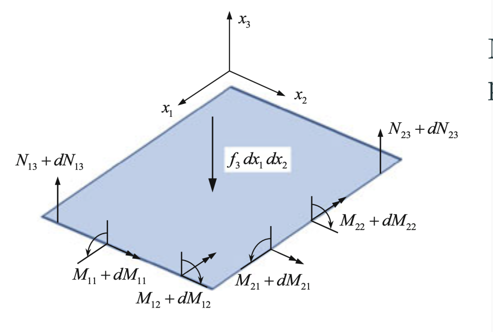
Distributed normal and shear stresses



Normal and shear stresses distributions through the thickness of the plate element:

- linear distributed normal stresses σ_{11} and σ_{22},
- linear distributed shear stresses σ_{12} and σ_{21},
- parabolic distributed shear stresses σ_{23} and σ_{13}.

Moments and shear forces



Moments and shear forces acting along the edge of the plate:

- bending moments M_{11} and M_{22},
- twisting moment M_{12},
- shear forces N_{13} and N_{23}.

$$\mathbf{M} = \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\, \boldsymbol{\sigma}\, dx_3 = -\boldsymbol{C}\boldsymbol{\nabla}_k u_3 \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3^2\, dx_3 = -\frac{h^3}{12}\boldsymbol{C}\boldsymbol{\nabla}_k u_3$$

Dynamic equilibrium equation

- Equilibrium condition for the vertical forces:

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + f_3 - \rho h \ddot{u}_3 = 0$$

- Equilibrium of moments:

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} = 0$$

$$\frac{\partial M_{22}}{\partial x_2} + \frac{\partial M_{12}}{\partial x_1} - N_{23} = 0$$

In matrix form:

$$\boldsymbol{\nabla}_k^T \mathbf{M} + f_3 = \rho h \ddot{u}_3$$

Strong form for Kirchhoff-Love plate bending

Let $\Omega = [-a, a] \times [-b, b]$. Find the transverse displacement $u_3 \in C^4(\Omega \times [0, T])$ such that

$$\frac{h^3}{12} \boldsymbol{\nabla}_k^T \boldsymbol{C} \boldsymbol{\nabla}_k u_3 + \rho h \ddot{u}_3 = f_3 \qquad \text{on } \Omega \times]0, T[\tag{1}$$

boundary conditions (simply supported):

$$u_3 = 0 \qquad \text{in } \partial\Omega \times]0, T[$$

$$\mathbf{M}_n = 0 \qquad \text{in } \partial\Omega \times]0, T[$$

initial conditions:

$$u_3(\cdot, 0) = u_0 \quad \text{in } \Omega$$

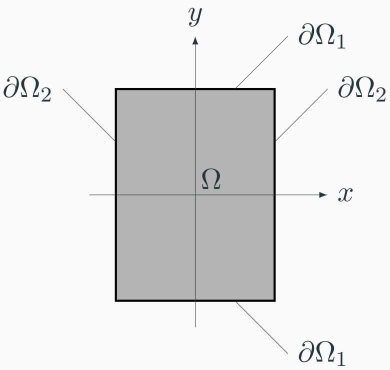
$$\dot{u}_3(\cdot, 0) = v_0 \quad \text{in } \Omega$$

In case of isotropic material equation (1) reduces to

$$D\left(\frac{\partial^4 u_3}{\partial x_1^4} + 2\frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u_3}{\partial x_2^4}\right) + \rho h \ddot{u}_3 = f_3$$

where $D = Eh^3/(12(1-\nu^2))$.

Approximated boundary conditions



Simply supported on all 4 edges:

- No vertical displacement:

$$u_3 = 0 \qquad \text{in } \partial\Omega \times]0, T[$$

- No moment resistance (free to rotate):

$$M_{11} = -D(\partial_x \varphi_2 + \nu \partial_y \varphi_1) = 0 \qquad \text{in } \partial\Omega_1 \times]0, T[$$
$$M_{22} = -D(\nu \partial_x \varphi_2 + \partial_y \varphi_1) = 0 \qquad \text{in } \partial\Omega_2 \times]0, T[$$

These conditions are replaced by the approximated conditions:

$$\varphi_2 = -\partial_x u_3 = 0 \qquad \text{in } \partial\Omega_1 \times]0, T[$$
$$\varphi_1 = \partial_y u_3 = 0 \qquad \text{in } \partial\Omega_2 \times]0, T[$$

Weak form for Kirchhoff-Love plate bending

The weak form consists of finding the transverse displacement $u_3 \in \mathcal{U}$ such that the following equation is satisfied for every $\delta u_3 \in \mathcal{V}$:

$$\frac{h^3}{12}\int_\Omega \boldsymbol{\nabla}_k u_3 \mathbf{C}\, \boldsymbol{\nabla}_k \delta u_3 \, d\Omega + \int_\Omega \rho h \, \ddot{u}_3 \, \delta u_3 \, d\Omega = \int_\Omega f_3 \, \delta u_3 \, d\Omega$$

$$\mathcal{U} = \left\{ u_3(\cdot, t) \in H^2(\Omega) \mid u_3 = 0 \text{ in } \partial\Omega \times]0, T[\right\}$$
$$\mathcal{V} = \left\{ \delta u_3 \in H^2(\Omega) \mid \delta u_3 = 0 \text{ in } \partial\Omega \right\}$$

Shell element

Shell element formulations

There are at least two methods of formulating shell elements:

- Combining a membrane element with a plate bending element to for a flat shell element.
  $\rightarrow$ Governing equation is the Kirchhoff plate bending equation:

$$\frac{h^3}{12}\boldsymbol{\nabla}_k^T \boldsymbol{C} \boldsymbol{\nabla}_k u_3 + \rho h \ddot{u}_3 = f_3 \qquad \text{on } \Omega \times]0, T[$$

- Deriving a curved element which is a degenerate solid element to form a thick shell element.
  $\rightarrow$ Governing equation is the elastodynamic equilibrium equation:

$$\nabla^T \overline{\mathbf{C}} \nabla \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}} \qquad \text{on } \Omega \times]0, T[$$

where Ω is characterized by the degenerate hypothesis that one dimension (suppose ξ_3) is significantly smaller than the other two.

Elasticity, linearity and isotropic hypothesis

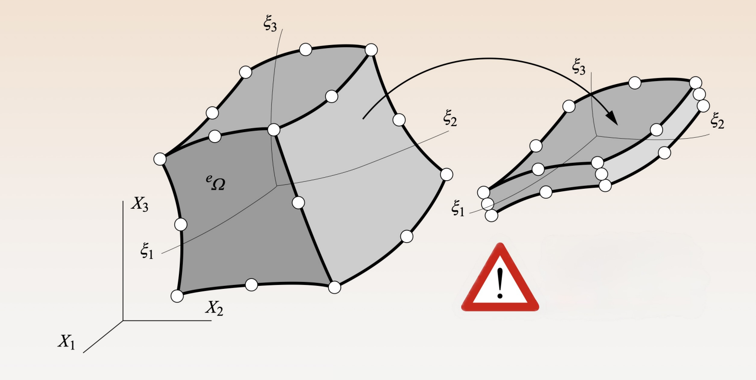
- Linear strain-displacement relationship: $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$ with $\varepsilon_{33} = 0$

$$\nabla = \begin{bmatrix} \partial_{x_1} & 0 & 0 \\ 0 & \partial_{x_2} & 0 \\ 0 & 0 & \partial_{x_3} \\ 0 & \partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & \partial_{x_1} \\ \partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix}$$

- Generalized Hooke's law: $\boldsymbol{\sigma} = \overline{\mathbf{C}}\boldsymbol{\varepsilon}$:

$$\overline{\mathbf{C}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k(1-\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & k(1-\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k(1-\nu)/2 \end{bmatrix}$$

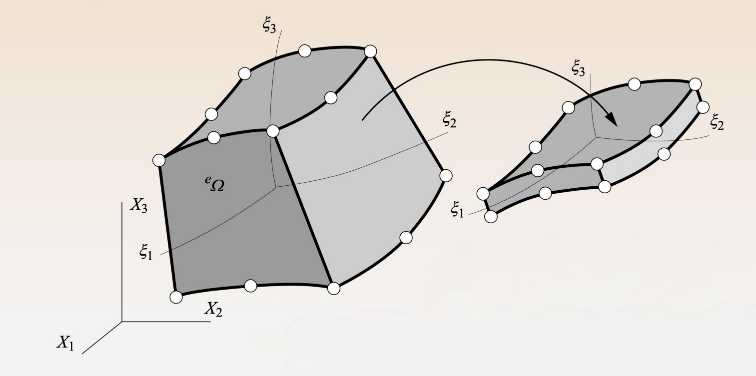
Degenerate solid (3d) finite elements



Rectilinearity of the normal vectors assumption is not respected

(Credit:(G))

Degenerate solid (3d) finite elements



Poor conditioning: excessive deformation

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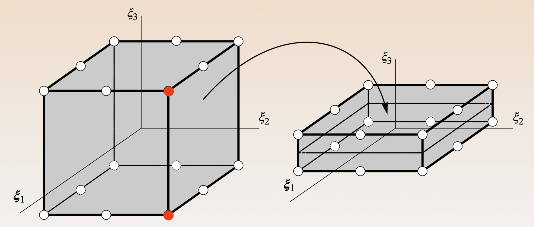
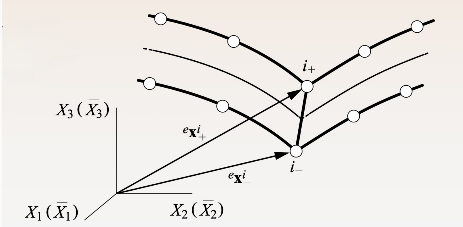
Coordinate transformation

Finite solid element with linear edges along the ξ_3 direction.



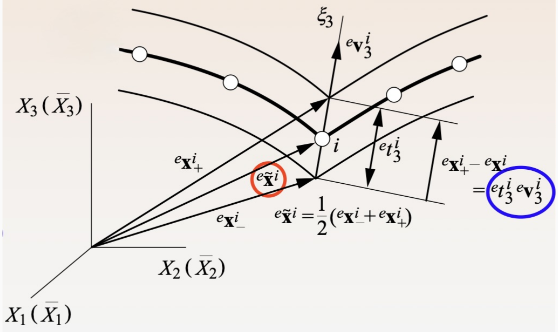


Nodes only on top and bottom faces in pairs. Let ${}^eq = 16$ the total number of nodes.

$$ {}^eT : \mathbf{x}(\boldsymbol{\xi}) = \sum_{i=1}^{{}^eq} {}^ah_i(\xi_1, \xi_2, \xi_3)\, {}^e\mathbf{x}^i = \sum_{i=1}^{{}^eq/2} {}^ah_i(\xi_1, \xi_2) \left[\frac{1-\xi_3}{2}\, {}^e\mathbf{x}^i_- + \frac{1+\xi_3}{2}\, {}^e\mathbf{x}^i_+ \right] $$

(Credit:(G))

Coordinate transformation



Let ${}^e p = {}^e q/2$ the total number of nodes on the shell midsurface.

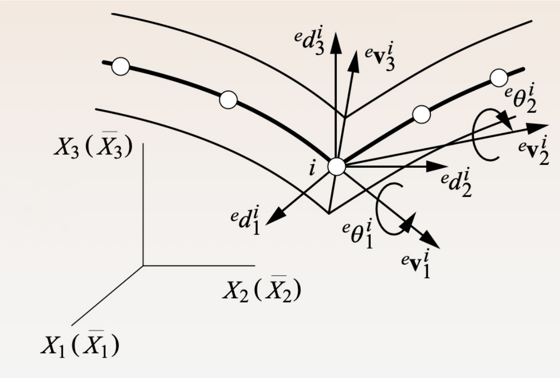
- ${}^e\tilde{\mathbf{x}}^i$ coordinate of node i on the midsurface of ${}^e\Omega$
- ${}^e t_3^i$ thickness at node i
- ${}^e\mathbf{v}_3^i$ normal vector to the midsurface at node i

$$
\begin{aligned}
{}^e T : \mathbf{x}(\boldsymbol{\xi}) &= \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left[\frac{1}{2}({}^e\mathbf{x}_-^i + {}^e\mathbf{x}_+^i) + \frac{1}{2}\xi_3({}^e\mathbf{x}_-^i - {}^e\mathbf{x}_+^i) \right] \\
&= \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left[{}^e\tilde{\mathbf{x}}^i + \frac{1}{2}\xi_3\, {}^e t_3^i\, {}^e\mathbf{v}_3^i \right]
\end{aligned}
$$

Approximate displacements

Analogy with transformation of coordinates:

$$\mathbf{u}^h(\boldsymbol{\xi}) = \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left[{}^e\mathbf{d}^i + \frac{1}{2}\xi_3 {}^e t_3^i \left(-{}^e\theta_1^i \, {}^e\mathbf{v}_2^i + {}^e\theta_2^i \, {}^e\mathbf{v}_1^i \right) \right]$$



$${}^e\mathbf{d}^i = [{}^e d_1^i, {}^e d_2^i, {}^e d_3^i]^T$$

- ${}^e d_1^i, {}^e d_2^i, {}^e d_3^i$ displacements of node i, oriented along the global axis.
- ${}^e \theta_1^i, {}^e \theta_2^i$ rotations of node i, oriented along the local axis.

Construction of local vectors

- We have:

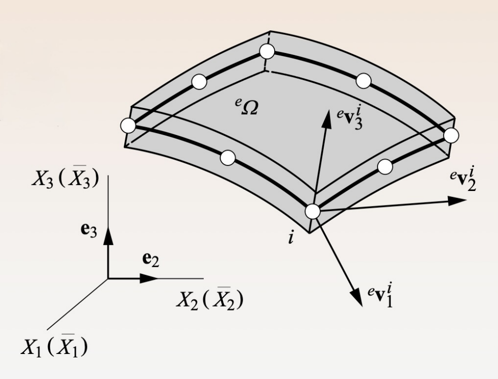
$$ {}^{e}\mathbf{v}_3^i = {}^{e}\mathbf{x}_-^i - {}^{e}\mathbf{x}_+^i $$

  the normal vector to the midsurface at node i.

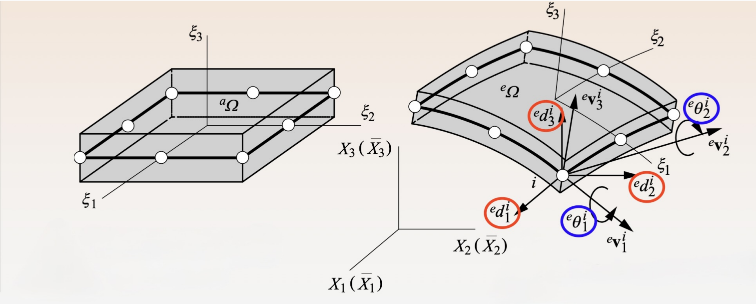
- We define the local vectors:

$$ {}^{e}\mathbf{v}_1^i = \begin{cases} \mathbf{e}_2 \wedge {}^{e}\mathbf{v}_3^i & \text{if } {}^{e}\mathbf{v}_3^i \neq \pm\mathbf{e}_2 \\ \pm\mathbf{e}_3 & \text{if } {}^{e}\mathbf{v}_3^i = \pm\mathbf{e}_2 \end{cases} $$

$$ {}^{e}\mathbf{v}_2^i = {}^{e}\mathbf{v}_3^i \wedge {}^{e}\mathbf{v}_1^i $$



Shell element



Shell elements:

- have 5 DOFs per node, no rotation $^e\theta_3^i$.
- lead to huge computational time savings since allow modeling with fewer mesh elements.
- less prone to negative Jacobian errors which might occur when using extremely thin 3d solid elements.

Shape functions matrix

$$\mathbf{u}^h(\boldsymbol{\xi}) = \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left[{}^e\mathbf{d}^i + \frac{1}{2}\xi_3 {}^e t_3^i \left(-{}^e\theta_1^i \, {}^e\mathbf{v}_2^i + {}^e\theta_2^i \, {}^e\mathbf{v}_1^i \right) \right]$$

$$= \sum_{i=1}^{{}^e p} {}^a\mathbf{H}_i(\boldsymbol{\xi}) {}^e\mathbf{q}^i(t)$$

$$= \sum_{i=1}^{{}^e p} \underbrace{\left[\begin{array}{ccc|cc} {}^a h_i & 0 & 0 & -\frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{21}^i & \frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{11}^i \\ 0 & {}^a h_i & 0 & -\frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, v_{22}^i & \frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{12}^i \\ 0 & 0 & {}^a h_i & -\frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{23}^i & \frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{13}^i \end{array} \right]}_{{}^a\mathbf{H}_i} \underbrace{\begin{bmatrix} {}^e d_1^i \\ {}^e d_2^i \\ {}^e d_3^i \\ {}^e \theta_1^i \\ {}^e \theta_2^i \end{bmatrix}}_{{}^e\mathbf{q}^i}$$

Deformation matrix

$$
\begin{aligned}
{}^{a}\mathbf{B}_i =& \nabla {}^{a}\mathbf{H}_i \\
=& \begin{bmatrix}
\frac{\partial {}^{e}h_i}{\partial x_1} & 0 & 0 & {}^{e}g^{i}_{11} & {}^{e}f^{i}_{11} \\
0 & \frac{\partial {}^{e}h_i}{\partial x_2} & 0 & {}^{e}g^{i}_{22} & {}^{e}f^{i}_{22} \\
0 & 0 & \frac{\partial {}^{e}h_i}{\partial x_3} & {}^{e}g^{i}_{33} & {}^{e}f^{i}_{33} \\
0 & \frac{\partial {}^{e}h_i}{\partial x_3} & \frac{\partial {}^{e}h_i}{\partial x_2} & {}^{e}g^{i}_{23} + {}^{e}g^{i}_{32} & {}^{e}f^{i}_{23} + {}^{e}f^{i}_{32} \\
\frac{\partial {}^{e}h_i}{\partial x_3} & 0 & \frac{\partial {}^{e}h_i}{\partial x_1} & {}^{e}g^{i}_{31} + {}^{e}g^{i}_{13} & {}^{e}f^{i}_{31} + {}^{e}f^{i}_{13} \\
\frac{\partial {}^{e}h_i}{\partial x_2} & \frac{\partial {}^{e}h_i}{\partial x_1} & 0 & {}^{e}g^{i}_{12} + {}^{e}g^{i}_{21} & {}^{e}f^{i}_{12} + {}^{e}f^{i}_{21}
\end{bmatrix} \qquad (i = 1, 2, \ldots, {}^{e}p)
\end{aligned}
$$

Derivative with respect to global variables:

$$\frac{\partial^e h_i}{\partial x_k} = \frac{\partial^a h_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_k} + \frac{\partial^a h_i}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_k} = {}^e J_{k1}^{-1}\frac{\partial^a h_i}{\partial \xi_1} + {}^e J_{k2}^{-1}\frac{\partial^a h_i}{\partial \xi_2}$$

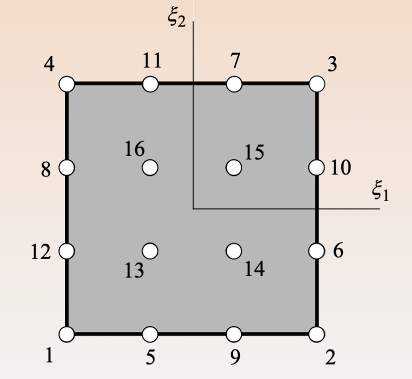
and

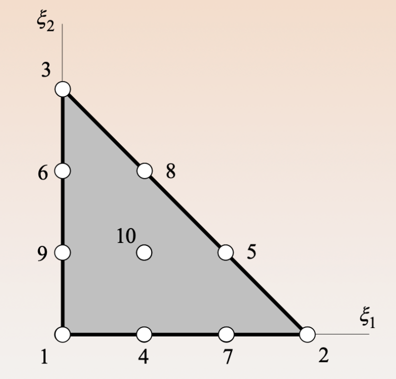
$${}^e f_{jk}^i = \frac{1}{2}\,{}^e t_3^i v_{1j}^i \frac{\partial(\xi_3\,{}^e h_i)}{\partial x_k},$$

$${}^e g_{jk}^i = -\frac{1}{2}\,{}^e t_3^i v_{2j}^i \frac{\partial(\xi_3\,{}^e h_i)}{\partial x_k}$$

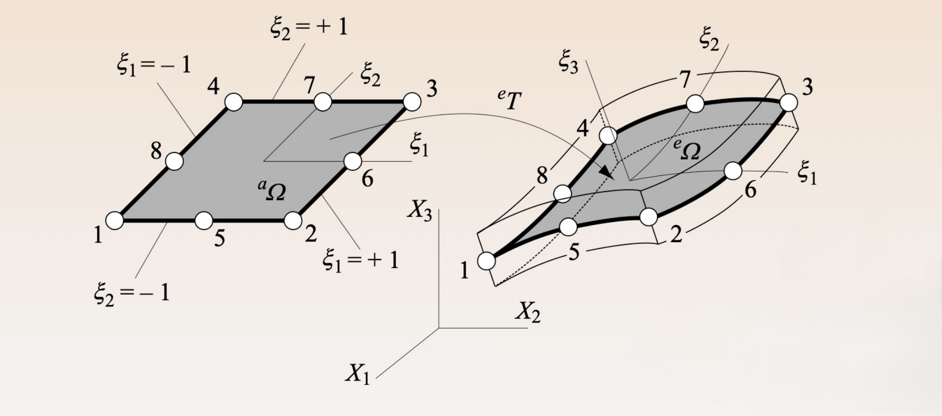
$$\frac{\partial(\xi_3\,{}^e h_i)}{\partial x_k} = \xi_3 \frac{\partial^e h_i}{\partial x_k} + {}^a h_i \frac{\partial \xi_3}{\partial x_k} = \xi_3\left({}^e J_{k1}^{-1}\frac{\partial^a h_i}{\partial \xi_1} + {}^e J_{k2}^{-1}\frac{\partial^a h_i}{\partial \xi_2}\right) + {}^e J_{k3}^{-1}\,{}^a h_i$$

Examples: quadrangular and triangular shell elements

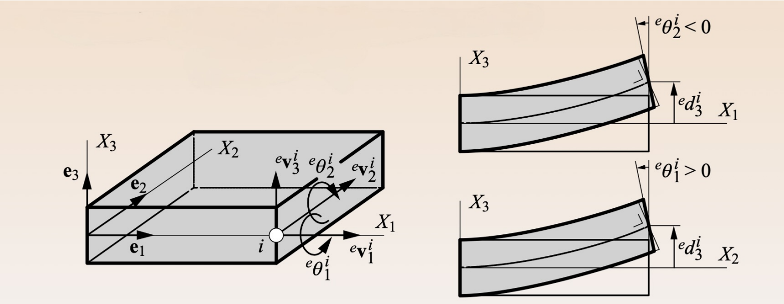




Example: 8 nodes quadrangular shell element



Example: flat shell element



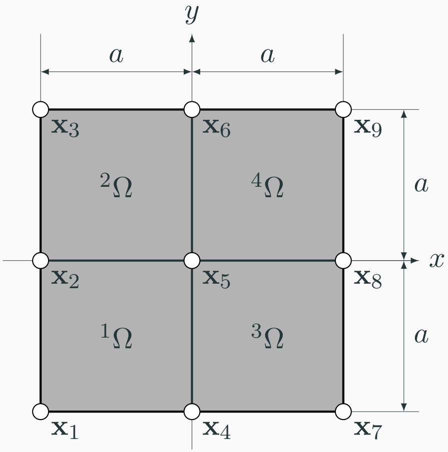
$$\mathbf{x}(\boldsymbol{\xi}) = \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left(\begin{bmatrix} {}^e x^i \\ {}^e y^i \\ 0 \end{bmatrix} + \frac{1}{2} \xi_3 {}^e t_3^i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\mathbf{u}^h(\boldsymbol{\xi}) = \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left(\begin{bmatrix} 0 \\ 0 \\ {}^e d_3^i \end{bmatrix} + \frac{1}{2} \xi_3 {}^e t_3^i \left(-{}^e\theta_1^i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + {}^e\theta_2^i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \right)$$

Example: modal analysis of simply supported plate

Simply supported isotropic plate

Discretization with 4 bilinear quadrilateral shell elements (4 nodes each).



- $2a$ length
- $2a$ height
- e thickness
- E Young's modulus
- ν Poisson's ratio
- ρ material density

Objective: determine the first natural frequency of the plate and compare it with the exact one.

Approximate displacements for flat shell elements

- Nodal displacements in the plane of the elements are suppressed:

$$ {}^{e}\mathbf{d}^{i} = [0, 0, {}^{e}d_3^i]^T $$

  Thus only 3 DOFs per node: ${}^{e}\mathbf{q}^{i} = [{}^{e}d_3^i,\ {}^{e}\theta_1^i,\ {}^{e}\theta_2^i]$.

- Local vectors are oriented along the principal axes of the shell:

$$ {}^{e}\mathbf{v}_1^i = [1, 0, 0]^T, \qquad {}^{e}\mathbf{v}_2^i = [0, 1, 0]^T, \qquad {}^{e}\mathbf{v}_3^i = [0, 0, 1]^T $$

- Local shape functions matrices:

$$ {}^{a}\mathbf{H}_i = \begin{bmatrix} 0 & 0 & \frac{1}{2}\xi_3 e\, {}^{a}h_i \\ 0 & -\frac{1}{2}\xi_3 e\, {}^{a}h_i & 0 \\ {}^{a}h_i & 0 & 0 \end{bmatrix} \qquad (i = 1, 2, 3, 4) $$

Coordinate transformation for $^1\Omega$

- Bilinear base functions for quadrilateral shell element:

$$\begin{aligned}
{}^a h_1(\xi_1,\xi_2) &= (1-\xi_1)(1-\xi_2)/4\\
{}^a h_2(\xi_1,\xi_2) &= (1+\xi_1)(1-\xi_2)/4\\
{}^a h_3(\xi_1,\xi_2) &= (1+\xi_1)(1+\xi_2)/4\\
{}^a h_4(\xi_1,\xi_2) &= (1-\xi_1)(1+\xi_2)/4
\end{aligned}$$

- Coordinates ${}^1\tilde{\mathbf{x}} = [[0,0,0],[a,0,0],[a,a,0],[0,a,0]]$

$$ {}^1T : \mathbf{x}(\boldsymbol{\xi}) = \sum_{i=1}^{4} {}^a h_i(\xi_1,\xi_2)\left({}^e\tilde{\mathbf{x}}^i + \frac{1}{2}\xi_3\, {}^e t_3^i\, {}^e\mathbf{v}_3^i \right) = \left[a\frac{1+\xi_1}{2},\ a\frac{1+\xi_2}{2}, e\frac{\xi_3}{2} \right]^T $$

- Jacobian matrix ${}^1J = \mathrm{diag}(a/2,\ a/2,\ e/2)$ and determinant ${}^1j = a^2 e/8$.

Local mass matrices

$$
{}^{1}\mathbf{M}_{ij} = \frac{\rho a^2 e}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} {}^{a}\mathbf{H}_i^{T\,a}\mathbf{H}_j d\xi_1 d\xi_2 d\xi_3
$$

$$
= \frac{\rho a^2 e}{4} \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix} {}^{a}h_i{}^{a}h_j & 0 & 0 \\ 0 & e^{2\,a}h_i{}^{a}h_j/12 & 0 \\ 0 & 0 & e^{2\,a}h_i{}^{a}h_j/12 \end{bmatrix} d\xi_1 d\xi_2
$$

Local mass matrix for ${}^{1}\Omega$ via exact integration:

$$
{}^{1}\mathbf{M} = \begin{bmatrix} {}^{1}\mathbf{M}_{11} & {}^{1}\mathbf{M}_{12} & {}^{1}\mathbf{M}_{13} & {}^{1}\mathbf{M}_{14} \\ & {}^{1}\mathbf{M}_{22} & {}^{1}\mathbf{M}_{23} & {}^{1}\mathbf{M}_{24} \\ & & {}^{1}\mathbf{M}_{33} & {}^{1}\mathbf{M}_{34} \\ sym. & & & {}^{1}\mathbf{M}_{44} \end{bmatrix}
$$

Local deformation matrices

$$
{}^{1}\mathbf{B}_i = \nabla^a \mathbf{H}_i = \begin{bmatrix} \partial_{x_1} & 0 & 0 \\ 0 & \partial_{x_2} & 0 \\ 0 & 0 & \partial_{x_3} \\ 0 & \partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & \partial_{x_1} \\ \partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{2}\xi_3 e\, {}^a h_i \\ 0 & -\frac{1}{2}\xi_3 e\, {}^a h_i & 0 \\ {}^a h_i & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & \frac{e}{a}\xi_3 \frac{\partial^a h_i}{\partial \xi_1} \\ 0 & -\frac{e}{a}\xi_3 \frac{\partial^a h_i}{\partial \xi_2} & 0 \\ 0 & 0 & 0 \\ \frac{2}{a}\frac{\partial^a h_i}{\partial \xi_2} & -{}^a h_i & 0 \\ \frac{2}{a}\frac{\partial^a h_i}{\partial \xi_1} & 0 & {}^a h_i \\ 0 & -\frac{e}{a}\xi_3 \frac{\partial^a h_i}{\partial \xi_1} & \frac{e}{a}\xi_3 \frac{\partial^a h_i}{\partial \xi_2} \end{bmatrix}
$$

Local stiffness matrices

$$ {}^1\mathbf{K}_{ij} = \frac{a^2 e}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} {}^1\mathbf{B}_i^T \overline{\mathbf{C}}\, {}^1\mathbf{B}_j d\xi_1 d\xi_2 d\xi_3 $$

Local stiffness matrix for ${}^1\Omega$ via exact integration

$$ {}^1\mathbf{K} = \begin{bmatrix} {}^1\mathbf{K}_{11} & {}^1\mathbf{K}_{12} & {}^1\mathbf{K}_{13} & {}^1\mathbf{K}_{14} \\ & {}^1\mathbf{K}_{22} & {}^1\mathbf{K}_{23} & {}^1\mathbf{M}_{24} \\ & & {}^1\mathbf{K}_{33} & {}^1\mathbf{K}_{34} \\ sym. & & & {}^1\mathbf{K}_{44} \end{bmatrix} $$

Assembly

- Since ${}^eJ = {}^1J$ and thus ${}^ej = {}^1j$ for every $e = 2, 3, 4$, we have
$$ {}^e\mathbf{K} = {}^1\mathbf{K} \qquad \text{and} \qquad {}^e\mathbf{M} = {}^1\mathbf{M} $$
- The assembly of the global stiffness $\mathbf{K}$ (27×27) and mass matrices $\mathbf{M}$ (27×27) can be performed using the connectivity table:

${}^e\Omega$	${}^1\Omega$	${}^2\Omega$	${}^3\Omega$	${}^4\Omega$
1	1	2	4	5
2	4	5	7	8
3	5	6	8	9
4	2	3	5	6

- The following 20 DOFs are constrained due the the fact that the plate is simply supported along its perimeter:

$d_3^1,\ \theta_1^1,\ \theta_2^1$ $\quad$ $d_3^3,\ \theta_1^3,\ \theta_2^3$ $\quad$ $d_3^7,\ \theta_1^7,\ \theta_2^7$ $\quad$ $d_3^9,\ \theta_1^9,\ \theta_2^9$

$d_3^2,\ \theta_1^2$ $\quad$ $d_3^4,\ \theta_2^4$ $\quad$ $d_3^6,\ \theta_2^6$ $\quad$ $d_3^8,\ \theta_1^8$

Modal analysis

- The semi-discrete weak form is a system of $27 - 20 = 7$ differential equations for the 7 free DOFs: $d_3^5,\ \theta_1^5,\ \theta_2^5$, and $\theta_2^2,\ \theta_1^4,\ \theta_1^6,\ \theta_2^8$

$$\mathbf{K}\mathbf{q}(t) + \mathbf{M}\ddot{\mathbf{q}}(t) = \mathbf{0}$$

- The first fundamental frequency $\omega_1 = \sqrt{\lambda_1}$ (in rad/s) can be computed solving the generalized eigenvalue problem: $(\mathbf{K} + \lambda\mathbf{M})\,\mathbf{p} = \mathbf{0}$.
- Assuming a thin plate $e/2a = 0.01$ and a shear coefficient $k = 5/6$ we obtain

$$\omega_1 = 0.6620\sqrt{E/(1-\nu^2)\rho a}$$

- From the analytical solution we obtain

$$\omega_1^{exact} = 0.0285\sqrt{E/(1-\nu^2)\rho a}$$

- Exact integration of transverse shear terms in the stiffness matrix $\mathbf{K}$ leads to *element locking*.

Selective integration

- Separate the transverse shear contributions terms ${}^e\mathbf{B}_i^\tau$ in the deformation matrix:

$$ {}^e\mathbf{B}_i = {}^e\mathbf{B}_i^\sigma + {}^e\mathbf{B}_i^\tau $$

- Split the stiffness matrix into flexural stiffness ${}^e\mathbf{K}_{ij}^\sigma$ and transverse shear stiffness ${}^e\mathbf{K}_{ij}^\tau$:

$$ {}^e\mathbf{K}_{ij} = {}^e\mathbf{K}_{ij}^\sigma + {}^e\mathbf{K}_{ij}^\tau = \int_\Omega {}^e\mathbf{B}_i^\sigma \overline{\mathbf{C}} {}^e\mathbf{B}_j^\sigma \, d\Omega + \int_\Omega {}^e\mathbf{B}_i^\tau \overline{\mathbf{C}} {}^e\mathbf{B}_j^\tau \, d\Omega $$

- Perform a **selective integration**: exact integration of bending contributions ${}^e\mathbf{K}_{ij}^\sigma$ and reduced integration (a single Gauss point located at the center of the element) for shear contributions ${}^e\mathbf{K}_{ij}^\tau$.
- Assuming a thin plate $e/2a = 0.01$ and a shear coefficient $k = 5/6$ with selective integration we obtain

$$ \omega_1 = 0.0383\sqrt{E/(1-\nu^2)\rho a} \qquad \text{(error} \approx 34\%) $$

Error estimates

Assuming a thin plate $e/2a = 0.01$ and a shear coefficient $k = 5/6$ with selective integration we obtain:

$$\omega_1 = 0.0383\sqrt{E/(1-\nu^2)\rho a} \qquad \text{(Rel. error} \approx 34\%)$$

Meshing	Integration	Elements	Rel. error
2×2	Exact	bilinear	$> 20'000\%$
2×2	Selective	bilinear	34%
4×4	Selective	bilinear	7.2%
1×1	Selective	biquadratic	6.2%
2×2	Selective	biquadratic	1.0%