Dynamic analysis of frames and grids

Classical structural elements

ME473 Dynamic finite element analysis of structures

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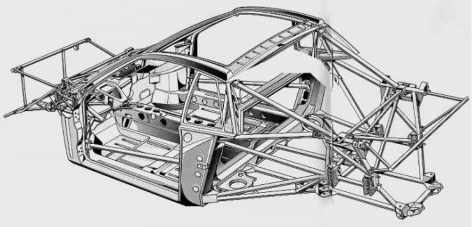
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation
3		FEM global	Project 1 statement
4		FEM local	
5		FEM local	Project 1 submission
6	Classical structural elements	Bars and trusses	Project 2 statement
7		Beams	
8		Frames and grids	

Summary

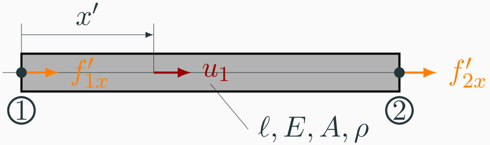
- Recap weeks 6 and 7
- Plane frames
- Plane grids
- Three-dimensional frames

Recommended readings

1. Logan, A first course in the finite element method, 6th ed. (chap. 5)
2. Paz and Leigh, Structural dynamics, 6th ed. (chap. 11, 12 and 13)
3. Ferreira and Fantuzzi, MATLAB Codes for Finite Element Analysis, 2nd ed. (chap. 7, 8 and 9)

Recap week 6 - bars and trusses

Non-oriented bar element



Differential equation governing the dynamics:

$$EA\partial^2_{x'x'}u_1(x',t) = \rho A\ddot{u}_1(x',t)$$

- Displacements approximation:

$$u_1^h(x',t) = \mathbf{H}(x')\mathbf{q}_{loc}(t) = \begin{bmatrix} h_1(x') & h_2(x') \end{bmatrix} \begin{bmatrix} q'_{1x}(t) \\ q'_{2x}(t) \end{bmatrix}$$

- Linear local shape functions:

$$h_1(x') = 1 - \frac{x'}{\ell} \qquad \text{and} \qquad h_2(x') = \frac{x'}{\ell}$$

- Semi-discrete weak form:

$$\boldsymbol{\delta}\mathbf{q}_{loc}^T\big(\mathbf{M}_{loc}\ddot{\mathbf{q}}_{loc}(t) + \mathbf{K}_{loc}\mathbf{q}_{loc}(t) - \mathbf{f}_{loc}(t)\big) = 0$$

Non-oriented bar element discretization

- Element stiffness matrix in local coordinates:

$$\mathbf{K}_{loc} = \int_0^\ell EA \, \frac{d\mathbf{H}}{dx'}^T \frac{d\mathbf{H}}{dx'} \, dx' = \int_0^\ell EA \begin{bmatrix} (h_1')^2 & h_1' h_2' \\ h_2' h_1' & (h_2')^2 \end{bmatrix} dx' = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Element consistent mass matrix in local coordinates:

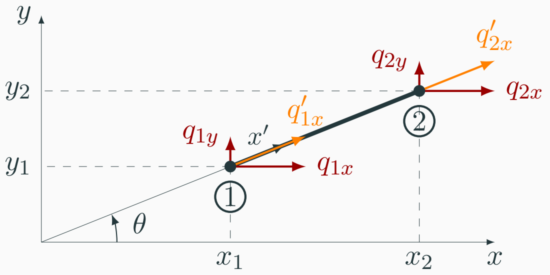
$$\mathbf{M}_{loc} = \int_0^\ell \rho A \, \mathbf{H}^T \mathbf{H} \, dx' = \int_0^\ell \rho A \begin{bmatrix} (h_1)^2 & h_1 h_2 \\ h_2 h_1 & (h_2)^2 \end{bmatrix} dx' = \frac{\rho A \ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Element applied loads vector in local coordinates:

$$\mathbf{f}_{loc}(t) = \begin{bmatrix} h_1(0) \\ h_2(0) \end{bmatrix} f_{1x}'(t) + \begin{bmatrix} h_1(\ell) \\ h_2(\ell) \end{bmatrix} f_{2x}'(t) = \begin{bmatrix} f_{1x}'(t) \\ f_{2x}'(t) \end{bmatrix}$$

Arbitrarly oriented bar element

- Displacements in local coordinates:
$$\mathbf{q}_{loc} = [q'_{1x}, q'_{2x}]^T$$
- Displacements in global coordinates:
$$\mathbf{q} = [q_{1x}, q_{1y}, q_{2x}, q_{2y}]^T$$
- **Relation between local and global displacements:**



$$\underbrace{\begin{bmatrix} q'_{1x} \\ q'_{2x} \end{bmatrix}}_{\mathbf{q}_{loc}} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \end{bmatrix}}_{\mathbf{q}}$$

Discretization of arbitrarily oriented bar

- Element stiffness matrix in global coordinates:

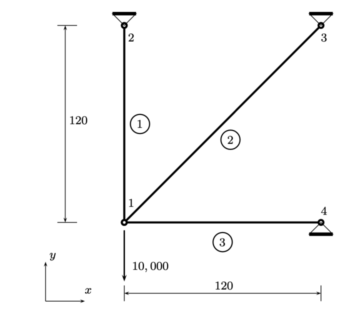
$$\mathbf{K} = \mathbf{T}^T \mathbf{K}_{loc} \mathbf{T} = \frac{EA}{\ell} \begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) & -\cos^2(\theta) & -\sin(\theta)\cos(\theta) \\ & \sin^2(\theta) & -\sin(\theta)\cos(\theta) & -\sin^2(\theta) \\ & & \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \text{Symm.} & & & \sin^2(\theta) \end{bmatrix}$$

- Element consistent mass matrix in global coordinates:

$$\mathbf{M} = \mathbf{T}^T \mathbf{M}_{loc} \mathbf{T} = \frac{\rho A \ell}{6} \begin{bmatrix} 2\cos^2(\theta) & 2\sin(\theta)\cos(\theta) & \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ & 2\sin^2(\theta) & \sin(\theta)\cos(\theta) & \sin^2(\theta) \\ & & 2\cos^2(\theta) & 2\sin(\theta)\cos(\theta) \\ \text{Symm.} & & & 2\sin^2(\theta) \end{bmatrix}$$

Illustrative example

Plane truss: structure composed of oriented bar elements that all lies in a common plane and are connected by frictionless pins.



Elements	Nodes	$^e\theta$	$^e\ell$
1	1, 2	90°	120 mm
2	1, 3	45°	$120\sqrt{2}$ mm
3	1, 4	0°	120 mm

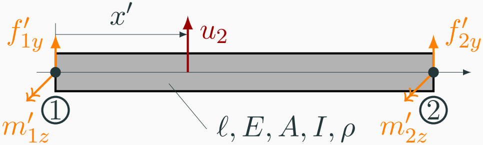
Elementary stiffness and mass matrices:

$$^e\mathbf{K} = {^e\mathbf{T}}\mathbf{K}_{loc}{^e\mathbf{T}} \qquad e = 1, 2, 3$$
$$^e\mathbf{M} = {^e\mathbf{T}}\mathbf{M}_{loc}{^e\mathbf{T}}$$

Global stiffness and mass matrices: $\mathbf{K} = \mathop{\mathsf{A}}_{e=1}^{3} {^e\mathbf{K}}$ and $\mathbf{M} = \mathop{\mathsf{A}}_{e=1}^{3} {^e\mathbf{M}}$.

Recap week 7 - planar beams

Planar beam element



Differential equation governing the dynamics:

$$\partial^2_{x'x'}\left(EI\partial^2_{x'x'}u_2(x',t)\right) + \rho A\ddot{u}_2(x',t) = 0$$

- Displacements approximation:

$$u_2^h(x',t) = \mathbf{H}(x')\mathbf{q}_{loc}(t) = \begin{bmatrix} h_1(x') & h_2(x') & h_3(x') & h_4(x') \end{bmatrix} \begin{bmatrix} q'_{1y}(t) \\ \phi'_{1z}(t) \\ q'_{2y}(t) \\ \phi'_{2z}(t) \end{bmatrix}$$

- Cubic local shape functions:

$$h_1(x') = 2(x'/\ell)^3 - 3(x'/\ell)^2 + 1 \qquad h_3(x') = 3(x'/\ell)^2 - 2(x'/\ell)^3$$

$$h_2(x') = x'(1 - x'/\ell)^2 \qquad h_4(x') = x'(x'/\ell)(x'/\ell - 1)$$

Discretization of beam

- Element stiffness matrix in local coordinates:

$$\mathbf{K}_{loc} = \int_0^\ell EI \, \frac{d^2\mathbf{H}}{(dx')^2}^T \frac{d^2\mathbf{H}}{(dx')^2} \, dx' = \frac{EI}{\ell^3} \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ & 4\ell^2 & -6\ell & 2\ell^2 \\ & & 12 & -6\ell \\ sym. & & & 4\ell^2 \end{bmatrix}$$

- Element consistent mass matrix in local coordinates:

$$\mathbf{M}_{loc} = \int_0^\ell \rho A \, \mathbf{H}^T \mathbf{H} \, dx' = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ & 4\ell^2 & 13\ell & -3\ell^2 \\ & & 156 & -22\ell \\ sym. & & & 4\ell^2 \end{bmatrix}$$

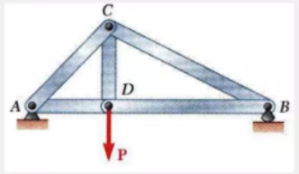
- Element applied loads vector in local coordinates:

$$\mathbf{f}_{loc}(t) = \begin{bmatrix} f'_{1y}(t) \\ m'_{1z}(t) \\ f'_{2y}(t) \\ m'_{2z}(t) \end{bmatrix}$$

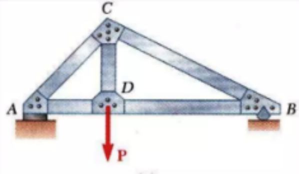
Plane frames

What is a plane frame?

- Structure composed of oriented beam elements, connected by welding, and carrying **transversal and axial forces** that all lies in a common plane.
- Both forces and moments can be transmitted between members.
- Loads are acting only in the common plane of the structure and they must be applied at the nodes or joints.



Plane truss

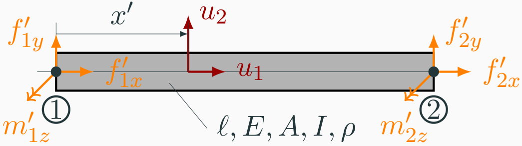


Plane frame

truss + beam = frame

Axial effects in beam element

The inclusion of axial forces in a flexural beam element requires a *superposition* of bar and beam elements:



Differential equations governing the dynamics:

$$\partial^2_{x'x'}\left(EI\partial^2_{x'x'}u_2(x',t)\right) + \rho A\ddot{u}_2(x',t) = 0$$
$$EA\partial^2_{x'x'}u_1(x',t) = \rho A\ddot{u}_1(x',t)$$

Axial effects are subsequently incorporated into the beam element formulation, unless specified otherwise.

Axial effects in beam element

The element now has three degrees of freedom per node: q'_{ix}, q'_{iy}, and ϕ'_{iz}.

- Displacements approximation:

$$u^h(x',t) = \mathbf{H}(x')\mathbf{q}_{loc}(t) = [h_1(x')\ h_2(x')\ h_3(x')\ h_4(x')\ h_5(x')\ h_6(x')] \begin{bmatrix} q'_{1x}(t) \\ q'_{1y}(t) \\ \phi'_{1z}(t) \\ q'_{2x}(t) \\ q'_{2y}(t) \\ \phi'_{2z}(t) \end{bmatrix}$$

- Local shape functions:

$$h_1(x') = 1 - x'/\ell \qquad h_4(x') = x'/\ell$$

$$h_2(x') = 2(x'/\ell)^3 - 3(x'/\ell)^2 + 1 \qquad h_5(x') = 3(x'/\ell)^2 - 2(x'/\ell)^3$$

$$h_3(x') = x'(1 - x'/\ell)^2 \qquad h_6(x') = x'(x'/\ell)(x'/\ell - 1)$$

Discretization of beam with axial effects

- Element stiffness matrix in local coordinates:

$$\mathbf{K}_{loc} = \frac{EI}{\ell^3} \begin{bmatrix} A\ell^2/I & 0 & 0 & -A\ell^2/I & 0 & 0 \\ & 12 & 6\ell & 0 & -12 & 6\ell \\ & & 4\ell^2 & 0 & -6\ell & 2\ell^2 \\ & & & A\ell^2/I & 0 & 0 \\ & & & & 12 & -6\ell \\ sym. & & & & & 4\ell^2 \end{bmatrix}$$

- Element consistent mass matrix in local coordinates:

$$\mathbf{M}_{loc} = \frac{\rho A \ell}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22\ell & 0 & 54 & -13\ell \\ & & 4\ell^2 & 0 & 13\ell & -3\ell^2 \\ & & & 70 & 0 & 0 \\ & & & & 156 & -22\ell \\ sym. & & & & & 4\ell^2 \end{bmatrix}$$

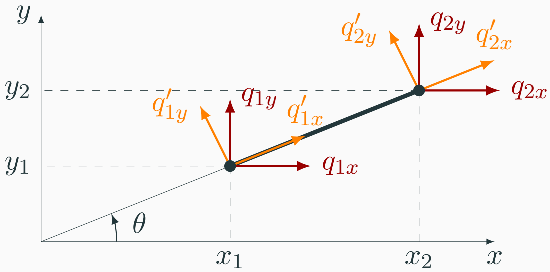
- Displacements in local coordinates:

  $$\mathbf{q}_{loc} = \begin{bmatrix} q'_{1x} & q'_{1y} & \phi'_{1z} & q'_{2x} & q'_{2y} & \phi'_{2z} \end{bmatrix}^T.$$

- Displacements in global coordinates:

  $$\mathbf{q} = \begin{bmatrix} q_{1x} & q_{1y} & \phi_{1z} & q_{2x} & q_{2y} & \phi_{2z} \end{bmatrix}^T.$$

- **Relation between local and global displacements:**



$$\underbrace{\begin{bmatrix} q'_{1x} \\ q'_{1y} \\ \phi'_{1z} \\ q'_{2x} \\ q'_{2y} \\ \phi'_{2z} \end{bmatrix}}_{\mathbf{q}_{loc}} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} q_{1x} \\ q_{1y} \\ \phi_{1z} \\ q_{2x} \\ q_{2y} \\ \phi_{2z} \end{bmatrix}}_{\mathbf{q}}$$

- Element stiffness matrix in global coordinates:

$$\mathbf{K} = \mathbf{T}^T\mathbf{K}_{loc}\mathbf{T} = \frac{E}{\ell}\begin{bmatrix} \frac{A\ell^3C^2+12IS^2}{\ell^2} & \frac{(-12I+A\ell^2)CS}{\ell^2} & \frac{-6IS}{\ell} & -\frac{A\ell^3C^2+12IS^2}{\ell^2} & \frac{(12I-A\ell^2)CS}{\ell^2} & \frac{-6IS}{\ell} \\ & \frac{12IC^2+A\ell^2S^2}{\ell^2} & \frac{6IC}{\ell} & \frac{(12I-A\ell^2)CS}{\ell^2} & -\frac{12IC^2+A\ell^2S^2}{\ell^2} & \frac{6IC}{\ell} \\ & & 4I & \frac{6IS}{\ell} & \frac{-6IC}{\ell} & 2I \\ & & & \frac{A\ell^3C^2+12IS^2}{\ell^2} & \frac{(-12I+A\ell^2)CS}{\ell^2} & \frac{6IS}{\ell} \\ & & & & \frac{12IC^2+A\ell^2S^2}{\ell^2} & \frac{-6IC}{\ell} \\ & & & & & 4I \end{bmatrix}$$

where $C = \cos(\theta)$ and $S = \sin(\theta)$.

- Element consistent mass matrix in global coordinates:

$$\mathbf{M} = \mathbf{T}^T\mathbf{M}_{loc}\mathbf{T}$$

Applied loads

- Element applied loads vector in local coordinates:

$$\mathbf{f}_{loc}(t) = \begin{bmatrix} f'_{1x}(t) \\ f'_{1y}(t) \\ m'_{1z}(t) \\ f'_{2x}(t) \\ f'_{2y}(t) \\ m'_{2z}(t) \end{bmatrix}$$

- Element applied loads vector in global coordinates:

$$\mathbf{f} = \mathbf{T}^T \mathbf{f}_{loc} = \begin{bmatrix} \cos(\theta) f'_{1x} + \sin(\theta) f'_{1y} \\ -\sin(\theta) f'_{1x} + \cos(\theta) f'_{1y} \\ m'_{1z} \\ \cos(\theta) f'_{2x} + \sin(\theta) f'_{2y} \\ -\sin(\theta) f'_{2x} + \cos(\theta) f'_{2y} \\ m'_{2z} \end{bmatrix}$$

Assembly of stiffness and mass matrices and loads vector

Given a 2d frame structure made of m oriented beams, n nodes, and 3 DOFs per node:

1. **Element quantities:**
   - For each beam e, compute the element quantities global coordinates:

$$
\begin{aligned}
{}^{e}\mathbf{K} &= {}^{e}\mathbf{T}^{T}\,{}^{e}\mathbf{K}_{loc}\,{}^{e}\mathbf{T} \\
{}^{e}\mathbf{M} &= {}^{e}\mathbf{T}^{T}\,{}^{e}\mathbf{M}_{loc}\,{}^{e}\mathbf{T} \\
{}^{e}\mathbf{f} &= {}^{e}\mathbf{T}^{T}\,{}^{e}\mathbf{f}_{loc}
\end{aligned}
$$

2. **Global assembly:**
   - Initialize global stiffness matrix $\mathbf{K}$ and global mass matrix $\mathbf{M}$ of size $3n \times 3n$,
   - Initialize global loads vector $\mathbf{f}$ of size $3n \times 1$,
   - Assemble each ${}^{e}\mathbf{K}$, ${}^{e}\mathbf{M}$ and ${}^{e}\mathbf{f}$ for $e = 1, \ldots, m$, into $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{f}$ respectively using element connectivity.

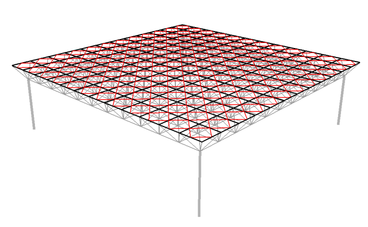
MATLAB example - a 2d frame in free vibration

▸ Go to Matlab Drive

Plane grids

What is a grid?

- A grid is a structure composed of oriented planar beams subjected to **perpendicular loading** that produced significant bending effects.
- Beams are connected by welding: both forces and moments can be transmitted between members.
- Axial effect of axial displacement is ignored for the moment.
- Very common type of structures used in to model floors, roofs, bridge deck systems, etc..



Degrees of freedom identification

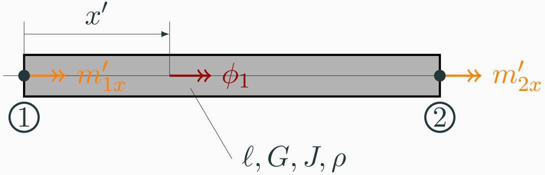
- **Planar frame under action of loads in the plane of the structure:**
  Components required to describe the displacements of joint i are:
  - q_{ix}, q_{iy} translations in the x and y directions respectively,
  - ϕ_{iz} (bending) rotation about the z axis.

- **Planar grids loaded perpendicularly to the plane of the structure:**
  Components required to describe the displacements of joint i are:
  - q_{iy} translation in the y direction,
  - ϕ_{ix}, (torsional rotation) ϕ_{iz} (bending rotation), rotations respectively about the x and z axes.

Shaft element

The development of a shaft finite element is very similar to the development of a bar finite element, where

- the axial displacement u_1 is replaced by the angular rotation ϕ_1,
- the axial nodal forces f'_{ix} are replaced by nodal torque m'_{ix},
- the element tensile stiffness AE/ℓ is replaced by the torsional stiffness GJ/ℓ.



- G shear modulus of the material
- J polar moment of inertia of the cross-section.
- ρ material density
- ℓ length
- ϕ_1 angular rotation
- x' (local) axial coordinate

Equation of motion for non-oriented shaft element

Differential equation governing the dynamics:

$$GJ\partial^2_{x'x'}\phi_1(x',t) = \rho J\ddot{\phi}_1(x',t)$$

- Angular rotation (twisting) approximation:

$$\phi_1^h(x',t) = \mathbf{H}(x')\mathbf{q}_{loc}(t) = \begin{bmatrix} h_1(x') & h_2(x') \end{bmatrix} \begin{bmatrix} \phi'_{1x}(t) \\ \phi'_{2x}(t) \end{bmatrix}$$

- Linear local shape functions:

$$h_1(x') = 1 - \frac{x'}{\ell} \qquad \text{and} \qquad h_2(x') = \frac{x'}{\ell}$$

- Semi-discrete weak form:

$$\boldsymbol{\delta}\mathbf{q}_{loc}^T\big(\mathbf{M}_{loc}\ddot{\mathbf{q}}_{loc}(t) + \mathbf{K}_{loc}\mathbf{q}_{loc}(t) - \mathbf{f}_{loc}(t)\big) = 0$$

Discretization of shaft

- Element stiffness matrix in local coordinates:

$$\mathbf{K}_{loc} = \int_0^\ell GJ \, \frac{d\mathbf{H}}{dx'}^T \frac{d\mathbf{H}}{dx'} \, dx' = \int_0^\ell GJ \begin{bmatrix} (h_1')^2 & h_1' h_2' \\ h_2' h_1' & (h_2')^2 \end{bmatrix} dx' = \frac{GJ}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Element consistent mass matrix in local coordinates:

$$\mathbf{M}_{loc} = \int_0^\ell \rho J \, \mathbf{H}^T \mathbf{H} \, dx' = \int_0^\ell \rho J \begin{bmatrix} (h_1)^2 & h_1 h_2 \\ h_2 h_1 & (h_2)^2 \end{bmatrix} dx' = \frac{\rho J \ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

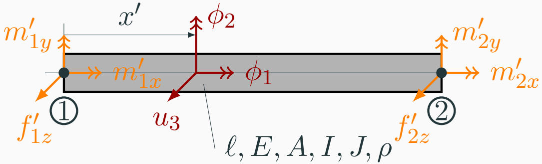
- Element applied loads vector in local coordinates:

$$\mathbf{f}_{loc}(t) = \begin{bmatrix} h_1(0) \\ h_2(0) \end{bmatrix} m_{1x}'(t) + \begin{bmatrix} h_1(\ell) \\ h_2(\ell) \end{bmatrix} m_{2x}'(t) = \begin{bmatrix} m_{1x}'(t) \\ m_{2x}'(t) \end{bmatrix}$$

shaft + beam = grid

Torsional effects in beam element

The inclusion of torsional stiffness in a flexural beam element, to model a typical element in a planar grid frame, requires a *superposition* of shaft and beam elements:



Differential equations governing the dynamics:

$$\partial^2_{x'x'}\left(EI\partial^2_{x'x'}u_3(x',t)\right) + \rho A\ddot{u}_3(x',t) = 0$$
$$GJ\partial^2_{x'x'}\phi_1(x',t) = \rho J\ddot{\phi}_1(x',t)$$

Torsional effects in beam element

Each grid element now has three degrees of freedom per node: ϕ'_{ix}, ϕ'_{iy} and q'_{iz}.

- Displacements approximation:

$$u^h(x',t) = \mathbf{H}(x')\mathbf{q}_{loc}(t) = [h_1(x')\ h_2(x')\ h_3(x')\ h_4(x')\ h_5(x')\ h_6(x')] \begin{bmatrix} \phi'_{1x}(t) \\ \phi'_{1y}(t) \\ q'_{1z}(t) \\ \phi'_{2x}(t) \\ \phi'_{2y}(t) \\ q'_{2z}(t) \end{bmatrix}$$

- Local shape functions:

$$h_1(x') = 1 - x'/\ell \qquad h_4(x') = x'/\ell$$

$$h_2(x') = x'(1 - x'/\ell)^2 \qquad h_5(x') = x'(x'/\ell)(x'/\ell - 1)$$

$$h_3(x') = 2(x'/\ell)^3 - 3(x'/\ell)^2 + 1 \qquad h_6(x') = 3(x'/\ell)^2 - 2(x'/\ell)^3$$

Discretization of beam with torsional effects

- Element stiffness matrix in local coordinates:

$$\mathbf{K}_{loc} = \frac{EI}{\ell^3} \begin{bmatrix} GJ\ell^2/EI & 0 & 0 & -GJ\ell^2/EI & 0 & 0 \\ & 4\ell^2 & -6\ell & 0 & 2\ell^2 & 6\ell \\ & & 12 & 0 & -6\ell & -12 \\ & & & GJ\ell^2/EI & 0 & 0 \\ & & & & 4\ell^2 & 6\ell \\ sym. & & & & & 12 \end{bmatrix}$$

- Element consistent mass matrix in local coordinates:

$$\mathbf{M}_{loc} = \frac{\rho A\ell}{420} \begin{bmatrix} 140J/A & 0 & 0 & 70J/A & 0 & 0 \\ 0 & 4\ell^2 & 22\ell & 0 & -3\ell^2 & 13\ell \\ & & 156 & 0 & -13\ell & 54 \\ & & & 140J/A & 0 & 0 \\ & & & & 4\ell^2 & -22\ell \\ sym. & & & & & 156 \end{bmatrix}$$

Arbitrarily oriented beam element with torsional effects

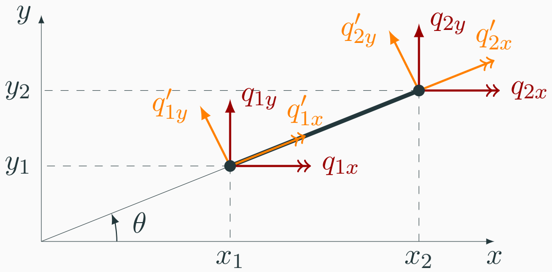
- Displacements in local coordinates:

  $$\mathbf{q}_{loc} = \begin{bmatrix}\phi'_{1x} & \phi'_{1y} & q'_{1z} & \phi'_{2x} & \phi'_{2y} & q'_{2z}\end{bmatrix}^T.$$

- Displacements in global coordinates:

  $$\mathbf{q} = \begin{bmatrix}\phi_{1x} & \phi_{1y} & q_{1z} & \phi_{2x} & \phi_{2y} & q_{2z}\end{bmatrix}^T.$$

- **Relation between local and global displacements:**



$$\underbrace{\begin{bmatrix}\phi'_{1x}\\ \phi'_{1y}\\ q'_{1z}\\ \phi'_{2x}\\ \phi'_{2y}\\ q'_{2z}\end{bmatrix}}_{\mathbf{q}_{loc}} = \underbrace{\begin{bmatrix}\cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0\\ -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0\\ 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix}\phi_{1x}\\ \phi_{1y}\\ q_{1z}\\ \phi_{2x}\\ \phi_{2y}\\ q_{2z}\end{bmatrix}}_{\mathbf{q}}$$

Discretization of arbitrarly oriented beam

- Element stiffness matrix in global coordinates:

$$\mathbf{K} = \mathbf{T}^T \mathbf{K}_{loc} \mathbf{T}$$

where $C = \cos(\theta)$ and $S = \sin(\theta)$.

- Element consistent mass matrix in global coordinates:

$$\mathbf{M} = \mathbf{T}^T \mathbf{M}_{loc} \mathbf{T}$$

- Element applied loads vector in global coordinates:

$$\mathbf{f} = \mathbf{T}^T \mathbf{f}_{loc}$$

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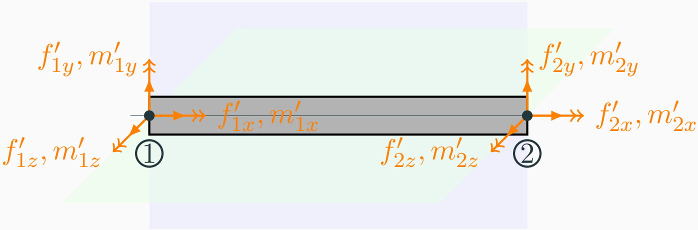
Three-dimensional beams

Example of a space beam structure

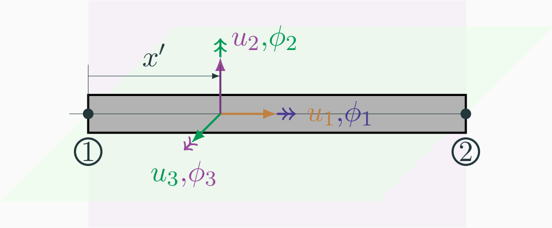
Credit: [N]



Three-dimensional beams are uniaxial (slender) element that can support:

- axial loads f'_{ix},
- torsional loads m'_{ix},
- bending in the $x' - y'$ plane: f'_{iy} and m'_{iz},
- bending in the $x' - z'$ plane: f'_{iz} and m'_{iy}.

Differential equations governing the dynamics



$$EA\partial^2_{x'x'}u_1(x',t) = \rho A\ddot{u}_1(x',t)$$

$$GJ\partial^2_{x'x'}\phi_1(x',t) = \rho J\ddot{\phi}_1(x',t)$$

$$\partial^2_{x'x'}\left(EI_z\partial^2_{x'x'}u_2(x',t)\right) + \rho A\ddot{u}_2(x',t) = 0$$

$$\partial^2_{x'x'}\left(EI_y\partial^2_{x'x'}u_3(x',t)\right) + \rho A\ddot{u}_3(x',t) = 0$$

I_y and I_z are the cross-sectional moments of inertia with respect to the axes y and z.

Displacements discretization

Total of six nodal displacements at each unconstrained joint:

- three translation components q'_{ix}, q'_{iy} and q'_{iz} along the x, y, z axes, and
- three rotational components about these axes ϕ'_{ix}, ϕ'_{iy} and ϕ'_{iz}.

$$u^h(x', t) = \mathbf{H}(x')\mathbf{q}_{loc}(t)$$

$$\mathbf{q}_{loc}(t) = \begin{bmatrix} q'_{1x}(t) \\ q'_{1y}(t) \\ q'_{1z}(t) \\ \phi'_{1x}(t) \\ \phi'_{1y}(t) \\ \phi'_{1z}(t) \\ q'_{2x}(t) \\ q'_{2y}(t) \\ q'_{2z}(t) \\ \phi'_{2x}(t) \\ \phi'_{2y}(t) \\ \phi'_{2z}(t) \end{bmatrix}$$

$$\begin{aligned}
h_1(x') &= 1 - x'/\ell & h_7(x') &= x'/\ell \\
h_2(x') &= 2(x'/\ell)^3 - 3(x'/\ell)^2 + 1 & h_8(x') &= 3(x'/\ell)^2 - 2(x'/\ell)^3 \\
h_3(x') &= 2(x'/\ell)^3 - 3(x'/\ell)^2 + 1 & h_9(x') &= 3(x'/\ell)^2 - 2(x'/\ell)^3 \\
h_4(x') &= 1 - x'/\ell & h_{10}(x') &= x'/\ell \\
h_5(x') &= x'(1 - x'/\ell)^2 & h_{11}(x') &= x'(x'/\ell)(x'/\ell - 1) \\
h_6(x') &= x'(1 - x'/\ell)^2 & h_{12}(x') &= x'(x'/\ell)(x'/\ell - 1)
\end{aligned}$$

Element stiffness matrix in local coordinates

The stiffness matrix for a three-dimensional uniform beam element is written by the superposition of the axial stiffness matrix, the torsional stiffness matrix and the flexural stiffness matrix:

$$\mathbf{K}_{loc} = \begin{bmatrix}
\frac{EA}{\ell} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{12EI_z}{\ell^3} & 0 & 0 & 0 & \frac{6EI_z}{\ell^2} & 0 & -\frac{12EI_z}{\ell^3} & 0 & 0 & 0 & \frac{6EI_z}{\ell^2} \\
 & & \frac{12EI_y}{\ell^3} & 0 & -\frac{6EI_y}{\ell^2} & 0 & 0 & 0 & -\frac{12EI_y}{\ell^3} & 0 & -\frac{6EI_y}{\ell^2} & 0 \\
 & & & \frac{GJ}{\ell} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{\ell} & 0 & 0 \\
 & & & & \frac{4EI_y}{\ell} & 0 & 0 & 0 & \frac{6EI_y}{\ell^2} & 0 & \frac{2EI_y}{\ell} & 0 \\
 & & & & & \frac{4EI_z}{\ell} & 0 & -\frac{6EI_z}{\ell^2} & 0 & 0 & 0 & \frac{2EI_z}{\ell} \\
 & & & & & & \frac{EA}{\ell} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & -\frac{12EI_z}{\ell^3} & 0 & 0 & 0 & -\frac{6EI_z}{\ell^2} \\
 & & & & & & & & \frac{12EI_y}{\ell^3} & 0 & \frac{6EI_y}{\ell^2} & 0 \\
 & & & & & & & & & \frac{GJ}{\ell} & 0 & 0 \\
 & & & & & & & & & & \frac{4EI_y}{\ell} & 0 \\
sym. & & & & & & & & & & & \frac{4EI_z}{\ell}
\end{bmatrix}$$

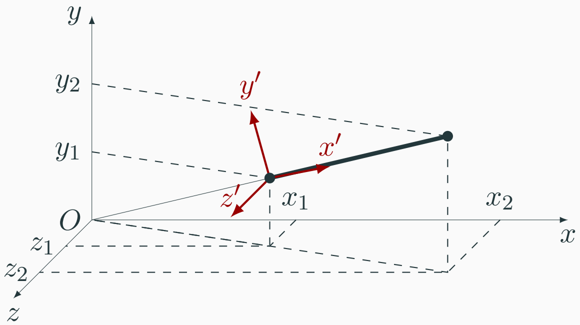
Element consistent mass matrix in local coordinates

The consistent mass matrix for a three-dimensional uniform beam element is written by the superposition of the axial mass matrix, the torsional mass matrix and the flexural mass matrix:

$$\mathbf{M}_{loc} = \frac{\rho A \ell}{420} \begin{bmatrix} 140 & 0 & 0 & 0 & 0 & 0 & 70 & 0 & 0 & 0 & 0 & 0 \\ & 156 & 0 & 0 & 0 & 22\ell & 0 & 54 & 0 & 0 & 0 & -13\ell \\ & & 156 & 0 & -22\ell & 0 & 0 & 0 & 54 & 0 & 13\ell & 0 \\ & & & \frac{140J}{A} & 0 & 0 & 0 & 0 & 0 & \frac{70J}{A} & 0 & 0 \\ & & & & 4\ell^2 & 0 & 0 & 0 & -13\ell & 0 & -3\ell^2 & 0 \\ & & & & & 4\ell^2 & 0 & 13\ell & 0 & 0 & 0 & -3\ell^2 \\ & & & & & & 140 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 156 & 0 & 0 & 0 & -22\ell \\ & & & & & & & & 156 & 0 & 22\ell & 0 \\ & & & & & & & & & \frac{140J}{A} & 0 & 0 \\ & & & & & & & & & & 4\ell^2 & 0 \\ sym. & & & & & & & & & & & 4\ell^2 \end{bmatrix}$$

Transformation of coordinates

- The stiffness and mass matrices are defined in the local coordinate system x', y' and z' fixed to the beam segment.
- To assemble the global stiffness and mass matrices, these local matrices must be transformed into the global coordinate system x, y, z.



Direction cosines

- The relationship between local and global coordinates is:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(x'x) & \cos(x'y) & \cos(x'z) \\ \cos(y'x) & \cos(y'y) & \cos(y'z) \\ \cos(z'x) & \cos(z'y) & \cos(z'z) \end{bmatrix}}_{\mathbf{T}'} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- The local x' is given by:

$$Ox' = [l_x, l_y, l_z]$$

where

$$l_x = \cos(x'x) = \frac{x_2 - x_1}{{}^e\ell}, \quad l_y = \cos(x'y) = \frac{y_2 - y_1}{{}^e\ell}, \quad l_z = \cos(x'z) = \frac{z_2 - z_1}{{}^e\ell},$$

$${}^e\ell = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Direction cosines

- The local y' axis is chosen so that it lies in the local $x' - y'$ plane and it is perpendicular to x':

$$Oy' = \frac{1}{d}\left[l_y, l_x, 0\right]$$

thus

$$\cos(y'x) = \frac{l_y}{d}, \quad \cos(y'y) = \frac{l_x}{d}, \quad \cos(y'z) = 0, \quad d = \sqrt{l_x^2 + l_y^2}.$$

- The local z' axis is chosen so that $Oz' = Ox' \times Oy'$:

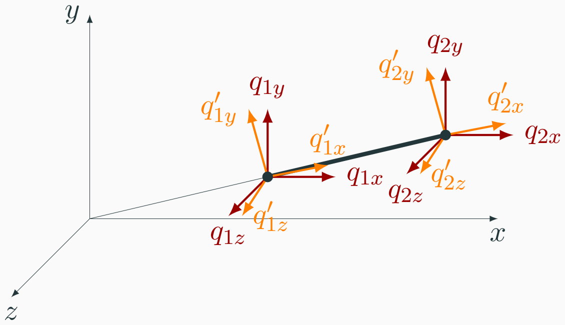
$$Oz' = \frac{1}{d}\left[-l_x l_z, -l_y l_z, 0\right]$$

thus

$$\cos(z'x) = -\frac{l_x l_z}{d}, \quad \cos(z'y) = -\frac{l_y l_z}{d}, \quad \cos(z'z) = d.$$

Global displacements for oriented three-dimensional beam

$$\underbrace{\begin{bmatrix} q'_{1x}(t) \\ q'_{1y}(t) \\ q'_{1z}(t) \\ \phi'_{1x}(t) \\ \phi'_{1y}(t) \\ \phi'_{1z}(t) \\ q'_{2x}(t) \\ q'_{2y}(t) \\ q'_{2z}(t) \\ \phi'_{2x}(t) \\ \phi'_{2y}(t) \\ \phi'_{2z}(t) \end{bmatrix}}_{\mathbf{q}_{loc}(t)} = \underbrace{\begin{bmatrix} \mathbf{T}' & & & \\ & \mathbf{T}' & & \\ & & \mathbf{T}' & \\ & & & \mathbf{T}' \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} q_{1x}(t) \\ q_{1y}(t) \\ q_{1z}(t) \\ \phi_{1x}(t) \\ \phi_{1y}(t) \\ \phi_{1z}(t) \\ q_{2x}(t) \\ q_{2y}(t) \\ q_{2z}(t) \\ \phi_{2x}(t) \\ \phi_{2y}(t) \\ \phi_{2z}(t) \end{bmatrix}}_{\mathbf{q}(t)}$$



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