# Dynamic analysis of beams

## Classical structural elements

ME473 Dynamic finite element analysis of structures

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# Where do we stand?

| Week | Module | Lecture topic | Mini-projects |
|---|---|---|---|
| 1 | Linear elastodynamics | Strong and weak forms | |
| 2 | | Galerkin method | Groups formation |
| 3 | | FEM global | Project 1 statement |
| 4 | | FEM local | |
| 5 | | FEM local | Project 1 submission |
| 6 | Classical structural elements | Bars and trusses | Project 2 statement |
| 7 | | Beams | |

## Summary

- Kinematics for Euler-Bernoulli beams
- Strong and weak forms for Euler-Bernoulli beams
- Stiffness and mass matrices
- Matlab example of a plane beam in free vibrations
- Geometric stiffness
- Matlab example of buckling
- Timoshenko beams

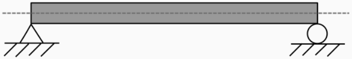
## Recommended readings

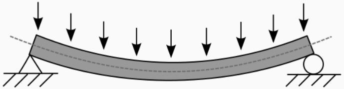
1. Logan, A first course in the finite element method, 6th ed. (chap. 4)
2. Paz and Leigh, Structural dynamics, 6th ed. (chap. 10)
3. Ferreira and Fantuzzi, MATLAB Codes for Finite Element Analysis, 2nd ed. (chap. 6 and 10)

# Euler-Bernoulli planar beams

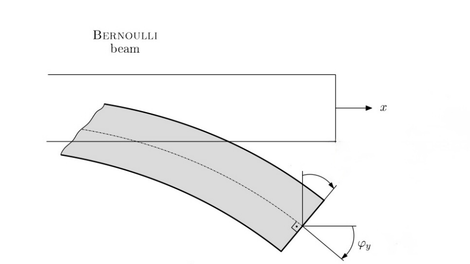
# What is a beam?

- Considered to be a uniaxial (slender) element: the longitudinal direction is sufficiently larger than the other two.
- Cross-section does not change along the element's length.
- Subjected to **transversal loading** that produced significant bending effects.
- Very common type of structures used in steel buildings, bridges, towers, etc...





# Euler-Bernoulli beam assumptions

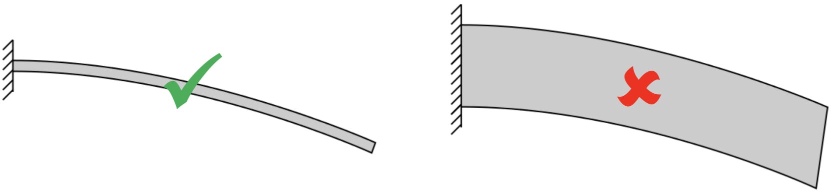



## Thin beam theory

- Plane cross section perpendicular to the longitudinal centroidal axis of the beam before bending occurs remains **plane** and **perpendicular** to the longitudinal axis after bending occurs.
- Shear deformations $\varepsilon_{12}$ of the plane cross section are neglected.
- The beam cross-section is infinitely rigid in its own plane.
- Reasonable model for slender structures made of isotropic materials.

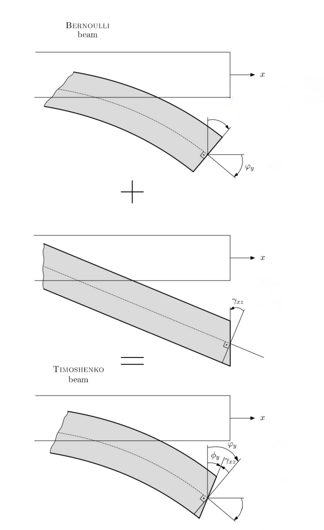
# Euler-Bernoulli beam assumptions

**Valid for:** slender beams ($h/\ell < 1/100$).



(Credit: Chatzi and Egger)

# Timoshenko beam assumptions

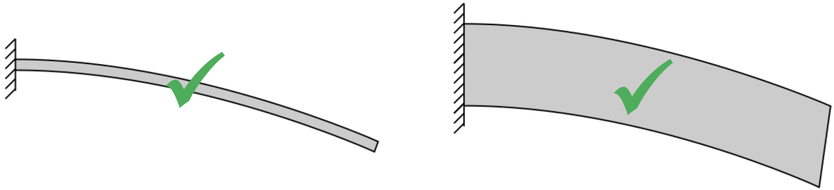



## Thick beam theory

- Plane cross section perpendicular to the longitudinal centroidal axis of the beam before bending occurs remains **plane** but **not necessarily perpendicular** to the longitudinal beam axis after beding occurs.
- Shear deformations $\varepsilon_{12}$ of the plane cross section are considered.
- Reasonable model for beams made of composite material that require the shear effect account.

# Timoshenko beam assumptions

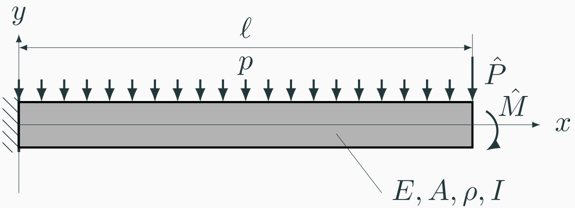
**Valid for:** slender beams ($h/\ell < 1/100$) and thick beams ($h/\ell > 1/10$).



(Credit: Chatzi and Egger)

# Kinematic assumptions for Euler-Bernoulli beam

1. The analysis will be restricted to the dynamic behavior of the beam in the $O(x, y)$ plane.
2. The beam cross-section remains plane after deformation.
3. Lines that are straight and perpendicular to the geometrical beam axis remain straight and perpendicular during deformation.




Model parameters:

- $A$ cross-sectional area
- $E$ Young's modulus
- $\rho$ material density
- $I$ moment of inertia
- $\ell$ length

Loads:

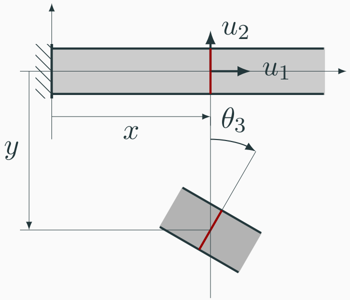
- $\hat{M}$ bending moment at free end
- $\hat{P}$ load at free end
- $p$ distributed transversal load

**Variables:**

- $u_1(x, t)$ axial displacement
- $u_2(x, t)$ transversal displacement

# Displacements field

Introduce an auxiliary variable $\theta_3(x,t)$ representing the total rotation of the section around the $Oz$ axis. Rotations are positive in clockwise direction.




- The first Euler-Bernoulli assumption implies

$$u_3 = 0.$$

- The second Euler-Bernoulli assumption implies

$$u_1 = -y\theta_3 \qquad \textit{(rotation-axial displ.)}$$

- The third Euler-Bernoulli assumption implies

$$\theta_3 = \partial_x u_2 \qquad \textit{(rotation-transversal displ.)}$$

**Only one unknown:** transversal displacement $u_2(x,t)$.

# Strain and stress

Substituting the displacement field $\mathbf{u}$ into $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$ yield to the strain-displacement relationship:

$$\varepsilon_{11} = \partial_x u_1 = -y\partial_{xx}^2 u_2 \qquad \varepsilon_{22} = \partial_y u_2 = 0$$

$$\varepsilon_{12} = \partial_x u_2 + \partial_y u_1 = \partial_x u_2 - \theta_3 = 0 \qquad \varepsilon_{33} = \varepsilon_{23} = \varepsilon_{13} = 0$$

Using the generalized Hooke's law for homogeneous and isotropic material, $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, is possible to write the stress field:

$$\sigma_{11} = (\lambda + 2G)\varepsilon_{11} \qquad \sigma_{22} = \lambda\varepsilon_{11}$$

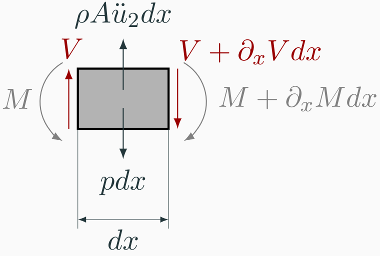
$$\sigma_{33} = \lambda\varepsilon_{11} \qquad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$$

where $\lambda = \frac{E}{(1+\nu)(1-2\nu)}$ and the shear modulus $G = \frac{E}{2(1+\nu)}$ are Lamé constants.

# Dynamic equilibrium equations

The theoretical stresses do not agree well with the experimental measurements.
⇒ **Additional assumptions regarding the stress field:**

$$\sigma_{11} = E\varepsilon_{11}, \qquad \text{and} \qquad \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0.$$




- Summation of the transverse forces:

$$\partial_x V + p = \rho A \ddot{u}_2$$

- Summation of the moments:

$$\partial_x M + V = 0$$

$$M = -\int_A y\sigma_{11}\, dA = -\int_A yE\varepsilon_{11}\, dA = EI\partial^2_{xx} u_2$$

# Strong form for transversal vibrations for Euler-Bernoulli beam

The strong form consists of finding the function $u_2 \in C^4([0,\ell] \times [0,T])$ such that the following equilibrium equation is satisfied:

$$\partial^2_{xx}\left(EI\partial^2_{xx}u_2\right) + \rho A\ddot{u}_2 = p$$

coupled with four boundary conditions:

$$u_2(0,t) = 0 \qquad \partial_x\left(-EI\partial^2_{xx}u_2\right)(\ell,t) = \hat{P}$$

$$\partial_x u_2(0,t) = 0 \qquad EI\partial^2_{xx}u_2(\ell,t) = \hat{M}$$

and two initial conditions: $u_2(x,0) = u_0(x)$ and $\dot{u}_2(x,0) = v_0(x)$.

# Weak form for transversal vibrations for Euler-Bernoulli beam

The weak form consists of finding the function $u_2 \in \mathcal{U}$ such that the following equation is satisfied for every $\delta u_2 \in \mathcal{V}$:

$$\int_0^\ell EI\, \partial_{xx}^2 u_2\, \partial_{xx}^2 \delta u_2\, dx + \int_0^\ell \rho A\, \ddot{u}_2\, \delta u_2\, dx = \int_0^\ell p\, \delta u_2\, dx + \hat{M}\, \delta u_2'(\ell) + \hat{P}\, \delta u_2(\ell)$$

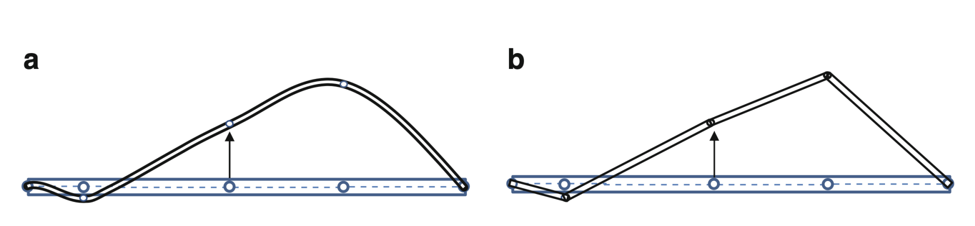
$$\mathcal{U} = \left\{ u_2(\cdot, t) \in H^2(]0,\ell[) \mid u_2(0,t) = \partial_x u_2(0,t) = 0\ \forall t \in ]0,T[ \right\}$$
$$\mathcal{V} = \left\{ \delta u_2 \in H^2(]0,\ell[) \mid \delta u_2(0) = \delta u_2'(0) = 0 \right\}$$

The Sobolev space $H^2(]0,\ell[)$ is the defined as:

$$H^2(]0,\ell[) = \left\{ f \in L^2(]0,\ell[) \mid \int_0^l \left(\partial_x f(x)\right)^2 dx < \infty,\ \int_0^l \left(\partial_{xx}^2 f(x)\right)^2 dx < \infty \right\}.$$

# Approximated transversal displacement

- Euler-Bernoulli theory states that both transversal displacement $u_2$ and rotation $\theta_3 = \partial_x u_2$ must be continuous within finite elements and in particular between elements.
- Shape functions that meet this requirement are said to have $C^1$ continuity.

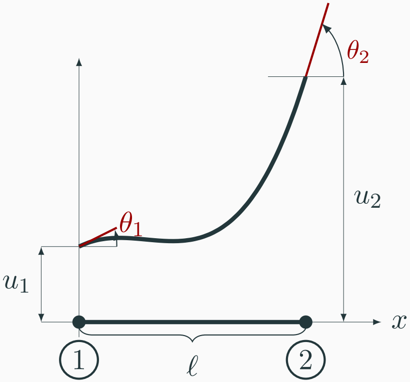



Cubic vs. linear elements. *Credit: [N]*

- Assume the approximated transversal displacement to be

$$u_2^h(x,t) = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t)$$

# Nodal degrees of freedom




**Parameters shifts:**

$$a_0,\ a_1,\ a_2,\ a_3, \quad \Rightarrow \quad u_1,\ u_2,\ \theta_1,\ \theta_2$$

A planar beam element has two DOFs per node:

- $u_1,\ u_2$: nodal displacements in the transverse direction.
- $\theta_1,\ \theta_2$: nodal rotations around the axis normal to the beam plane.

# Hermite $C^1$ shape functions

- Expressing the approximated displacement $u^h$ as a function of the nodal DOFs yield to:

$$u_2^h(x,t) = \mathbf{H}(x)\mathbf{q}(t) = [h_1(x),\ h_2(x),\ h_3(x),\ h_4(x)] \begin{bmatrix} u_1(t) \\ \theta_1(t) \\ u_2(t) \\ \theta_2(t) \end{bmatrix}$$

- **Cubic local shape functions:**

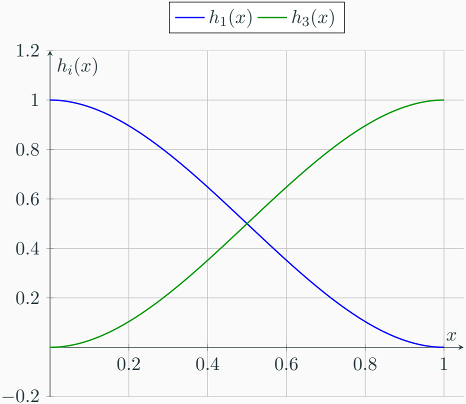
$$h_1(x) = 2(x/\ell)^3 - 3(x/\ell)^2 + 1 \qquad h_3(x) = 3(x/\ell)^2 - 2(x/\ell)^3$$

$$h_2(x) = x(1 - x/\ell)^2 \qquad h_4(x) = x(x/\ell)(x/\ell - 1)$$

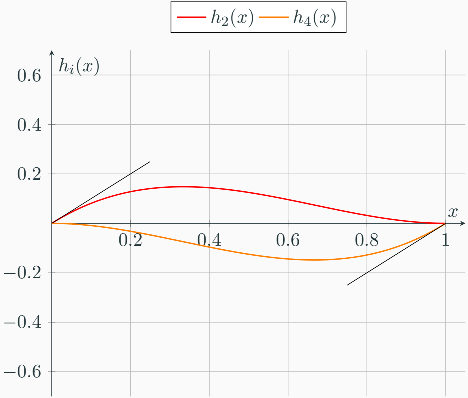
- Virtual displacement approximation follows the same logic:

$$\delta u^h(x) = \mathbf{H}(x)\boldsymbol{\delta}\mathbf{q} \qquad \text{where} \qquad \boldsymbol{\delta}\mathbf{q} = [\delta u_1,\ \delta\theta_1,\ \delta u_2,\ \delta\theta_2]^T$$

# Hermite $C^1$ shape functions visualizations




**Translational shape functions**




**Rotational shape functions**

# Element stiffness matrix

$$\mathbf{K} = \int_0^\ell EI\ \mathbf{B}^T\mathbf{B}\ dx$$

$$= \int_0^\ell EI \begin{bmatrix} (h_1'')^2 & h_1''h_2'' & h_1''h_3'' & h_1''h_4'' \\ & (h_2'')^2 & h_2''h_3'' & h_2''h_4'' \\ & & (h_3'')^2 & h_3''h_4'' \\ Symm. & & & (h_4'')^2 \end{bmatrix} dx$$

$$= \frac{EI}{\ell^3} \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ & 4\ell^2 & -6\ell & 2\ell^2 \\ & & 12 & -6\ell \\ Symm. & & & 4\ell^2 \end{bmatrix}$$

where the deformation matrix is

$$\mathbf{B} = \frac{d^2\mathbf{H}}{dx^2} = \begin{bmatrix} h_1''(x) & h_2''(x) & h_3''(x) & h_4''(x) \end{bmatrix}$$

# Element consistent mass matrix

$$\mathbf{M} = \int_0^\ell \rho A \; \mathbf{H}^T \mathbf{H} \; dx$$

$$= \int_0^\ell \rho A \begin{bmatrix} (h_1)^2 & h_1 h_2 & h_1 h_3 & h_1 h_4 \\ & (h_2)^2 & h_2 h_3 & h_2 h_4 \\ & & (h_3)^2 & h_3 h_4 \\ Symm. & & & (h_4)^2 \end{bmatrix} dx$$

$$= \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ & 4\ell^2 & 13\ell & -3\ell^2 \\ & & 156 & -22\ell \\ Symm. & & & 4\ell^2 \end{bmatrix}$$

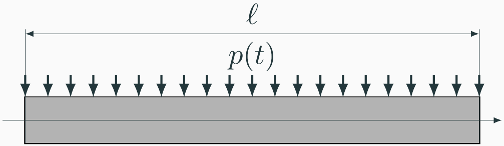
# Element lumped mass matrix

- Assumption that the mass of the structure is lumped at the nodal coordinates where translational displacements are defined.
- Inertial effect associated with any rotational degree of freedom is usually assumed to be zero.
- Recall that $\rho \cdot A$ is the mass per unit length along the beam, then the lumped mass matrix is defined as

$$\mathbf{M} = \frac{\rho A \ell}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Applied loads

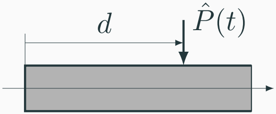
- **Constant distributed loads:**

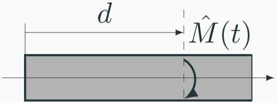



$$\mathbf{r} = \int_0^\ell p(t)\,\mathbf{H}^T\,dx = \frac{p(t)\ell}{2}\begin{bmatrix}1\\ \ell/6\\ 1\\ -\ell/6\end{bmatrix}$$

Uniform transverse loads can be replaced by two equivalent transverse nodes loads of value $(p\ell)/2$ and by two equivalent nodal moments of values $\pm p\ell^2/12$.

- **Concentrated loads:**




$$\mathbf{r}(t) = \hat{P}(t)\,\mathbf{H}^T(d)$$

$$\mathbf{r}(t) = \hat{M}(t)\,\frac{d\mathbf{H}}{dx}^T(d)$$

# Post-processing: approximation of bending moment and shear force

- The approximated bending moment along the beam element is

$$M^h(x,t) = EI\partial^2_{xx} u^h_2(x,t) = EI\,\mathbf{B}(x)\mathbf{q}(t).$$

- The approximated shear force along the beam element is

$$V^h(x,t) = -\partial_x M^h(x,t) = -EI\,\frac{d\mathbf{B}}{dx}(x)\mathbf{q}(t).$$

# MATLAB examples

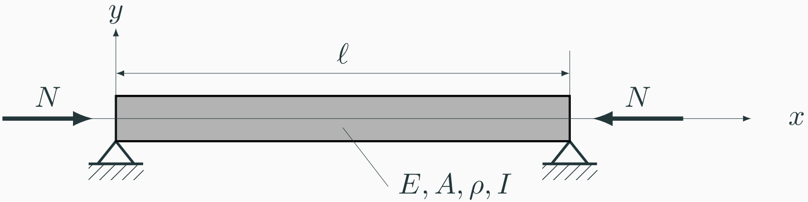
- A Cantilever beam subjected to a downward force
- A clamped-clamped beam in free vibration.

▸ Go to Matlab Drive

# Geometric stiffness

# Beam subjected to a distributed axial force

- Consider a beam element as used previously but now subjected to a distributed axial force $N$ per unit of length.

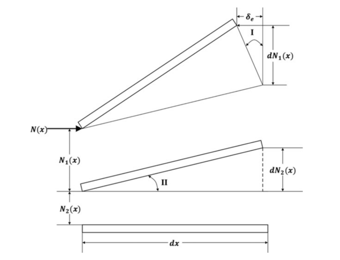



- Beam carrying large axial loads or undergoing large displacements have nonlinear behavior arising from the internal moments that are the product of the axial loads and the displacements transverse to the loads.
- The stiffness coefficients $k_{ij}$ must be modified by the presence of the axial force to the corresponding geometric stiffness coefficients $k_{ij}^g$.

# Geometric stiffness matrix

$k^g_{ij}$ is defined as the force corresponding to the nodal coordinate $i$ due to a unit displacement at coordinate $j$ and resulting for the axial force $N$.

**Calculation of $k^g_{12}$:** vertical force at node 1 due to a unit rotation $\theta_2 = 1$ caused by the axial force $N$. May be evaluated by the principle of virtual work:




$$dW = N(x)\delta_e$$

By similar triangles we have

$$\frac{\delta_e}{dh_1(x)} = \frac{dh_2(x)}{dx}$$

$$\delta_e = \frac{dh_1}{dx}(x)\frac{dh_2}{dx}(x)\,dx$$

$$k^g_{12} = \int_0^\ell N(x)h_1'(x)h_2'(x)\,dx$$

## Geometric stiffness matrix

- In general, any geometric stiffness coefficient may be expressed as

$$k_{ij}^g = \int_0^\ell N(x) h_i'(x) h_i'(x)\, dx$$

- We define the **consistent geometric stiffness matrix** as:

$$\mathbf{K}^g = \int_0^\ell N(x)\, \frac{d\mathbf{H}}{dx}^T \frac{d\mathbf{H}}{dx}\, dx$$

- When $N$ is constant along the beam length we obtain

$$\mathbf{K}^g = \frac{N}{30\ell} \begin{bmatrix} 36 & 3\ell & -36 & 3\ell \\ 3\ell & 4\ell^2 & -3\ell & -\ell^2 \\ -36 & -3\ell & 36 & -3\ell \\ 3\ell & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix}$$

# Example: stability of Bernoulli beam

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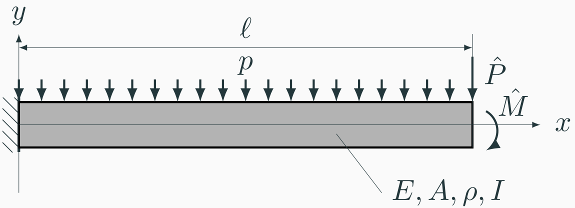
# Timoshenko planar beam

# Kinematic assumptions for Timoshenko beam

1. The analysis will be restricted to the dynamic behavior of the beam in the $O(x, y)$ plane.
2. The beam cross-section remains plane after deformation.
3. Lines that are straight and perpendicular to the geometrical beam axis remain straight but **not necessarily perpendicular** during deformation.




Model parameters:

- $A$ cross-sectional area
- $E$ Young's modulus
- $\rho$ material density
- $I$ moment of inertia
- $\ell$ length

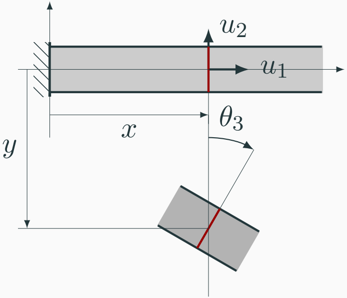
Loads:

- $\hat{M}$ bending moment at free end
- $\hat{P}$ load at free end
- $p$ distributed transversal load

**Variables:**

- $u_1(x, t)$ axial displacement
- $u_2(x, t)$ transversal displacement

Introduce an auxiliary variable $\theta_3(x,t)$ representing the total rotation of the section around the $Oz$ axis. Rotations are positive in clockwise direction.




- The first Timoshenko assumption implies

$$u_3 = 0.$$

- The second Timoshenko assumption implies

$$u_1 = -y\theta_3 \qquad \textit{(rotation-axial displ.)}$$

**Two unknowns:** transversal displacement $u_2(x,t)$ and $\theta_3(x,t)$ rotation of the section around the $Oz$ axis.

## Strain and stress

Substituting the displacement field $\mathbf{u}$ into $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$ yield to the strain-displacement relationship:

$$\varepsilon_{11} = \partial_x u_1 = -y\partial^2_{xx} u_2 \qquad \varepsilon_{22} = \partial_y u_2 = 0$$
$$\varepsilon_{12} = \partial_x u_2 + \partial_y u_1 = \partial_x u_2 - \theta_3 \qquad \varepsilon_{33} = \varepsilon_{23} = \varepsilon_{13} = 0$$

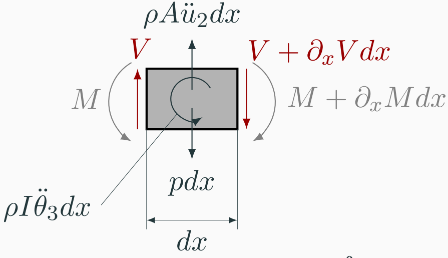
Using the generalized Hooke's law for homogeneous and isotropic material, $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, is possible to write the stress field:

$$\sigma_{11} = (\lambda + 2G)\varepsilon_{11} \qquad \sigma_{12} = 2G\varepsilon_{12}$$
$$\sigma_{22} = \sigma_{33} = \lambda\varepsilon_{11} \qquad \sigma_{13} = \sigma_{23} = 0$$

# Dynamic equilibrium equations

**Additional assumptions regarding the stress field:**

$$\sigma_{11} = E\varepsilon_{11}, \quad \sigma_{12} = 2G\varepsilon_{12} \qquad \text{and} \qquad \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0.$$




- Summation of the transverse forces:

$$\partial_x V + p = \rho A\ddot{u}_2$$

- Summation of the moments:

$$\partial_x M + V = \rho I\ddot{\theta}_3$$

$$V = \int_A \sigma_{12}\, dA = kGA\varepsilon_{12} = kGA(\partial_x u_2 - \theta_3)$$

$$M = -\int_A y\sigma_{11}\, dA = -\int_A yE\varepsilon_{11}\, dA = EI\partial_x\theta_3$$

## Strong form for transversal vibrations for Timoshenko beam

The strong form for transversal vibrations of beams consists of finding the functions $u_2 \in C^2([0,\ell] \times [0,T])$ and $\theta_3 \in C^2([0,\ell] \times [0,T])$ such that the following equilibrium equations, boundary and initial conditions are satisfied.

$$\partial_x \big( kGA(\partial_x u_2 - \theta_3) \big) + p = \rho A \ddot{u}_2$$

$$\partial_x \big( EI \partial_x \theta_3 \big) + kGA(\partial_x u_2 - \theta_3) = \rho I \ddot{\theta}_3$$

In matrix form:

$$\underbrace{\begin{pmatrix} \partial_x & 0 \\ 1 & \partial_x \end{pmatrix}}_{\nabla_\sigma^T} \underbrace{\begin{pmatrix} kGA & 0 \\ 0 & EI \end{pmatrix}}_{\mathbf{C}} \underbrace{\begin{pmatrix} \partial_x & -1 \\ 0 & \partial_x \end{pmatrix}}_{\nabla_u} \underbrace{\begin{pmatrix} u_2 \\ \theta_3 \end{pmatrix}}_{\mathbf{u}} + \underbrace{\begin{pmatrix} p \\ 0 \end{pmatrix}}_{\mathbf{f}} = \underbrace{\begin{pmatrix} \rho A & 0 \\ 0 & \rho I \end{pmatrix}}_{\mathbf{M}} \underbrace{\begin{pmatrix} \ddot{u}_2 \\ \ddot{\theta}_3 \end{pmatrix}}_{\ddot{\mathbf{u}}}$$

$$\nabla_\sigma^T \mathbf{C} \nabla_u \mathbf{u} + \mathbf{f} = \mathbf{M} \ddot{\mathbf{u}}$$

# Weak form for transversal vibrations for Timoshenko beam

The weak formulation of the problem consists in finding the solution $\mathbf{u} \in \mathcal{U}$ which satisfies the equation

$$\int_0^\ell (\nabla_u \boldsymbol{\delta}\mathbf{u})^{\mathrm{T}} \, \mathbf{C} \, (\nabla_u \mathbf{u}) \, \mathrm{d}x + \int_0^\ell \boldsymbol{\delta}\mathbf{u}^{\mathrm{T}} \mathbf{M} \ddot{\mathbf{u}} \, dx = \int_0^\ell \boldsymbol{\delta}\mathbf{u}^{\mathrm{T}} \mathbf{f} dx + \boldsymbol{\delta}\mathbf{u}^{\mathrm{T}}(\ell) \hat{\mathbf{f}} \quad \forall \boldsymbol{\delta}\mathbf{u} \in \mathcal{V}$$

where the function classes $\mathcal{U}$ and $\mathcal{V}$ are defined as follows

$$\mathcal{U} = \left\{ \mathbf{u} = \{u_2, \theta_3\}^{\mathrm{T}} \mid u_2(\cdot, t) \in H^1(]0,\ell[); \theta_3(\cdot, t) \in H^1(]0,\ell[); u_2(0,t) = \theta_3(0,t) = 0 \right\},$$

$$\mathcal{V} = \left\{ \boldsymbol{\delta}\mathbf{u} = \{\delta u_2, \delta\theta_3\}^{\mathrm{T}} \mid \delta u_2 \in H^1(]0,\ell[); \delta\theta_3 \in H^1(]0,\ell[); \delta u_2(0) = \delta\theta_3(0) = 0 \right\}.$$

# Displacements approximation

- We approximate the displacement using the ansatz:

$$\mathbf{u}^h(\mathbf{x}, t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t)$$
$$\delta\mathbf{u}^h(\mathbf{x}) = \mathbf{H}(\mathbf{x})\delta\mathbf{q}$$

- $\mathbf{H}(\mathbf{x})$ is a matrix of **linear shape functions**:

$$\mathbf{H} = \begin{bmatrix} h_1\mathbf{I} & h_2\mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 - x/\ell & 0 & x/\ell & 0 \\ 0 & 1 - x/\ell & 0 & x/\ell \end{bmatrix}$$

- $\mathbf{q}(t)$ is a vector of (*unknown*) time-dependent nodal displacements and $\boldsymbol{\delta}\mathbf{q}$ is a vector of constants:

$$\mathbf{q}(t) = \begin{bmatrix} u_2(t) \\ \theta_3(t) \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\delta}\mathbf{q} = \begin{bmatrix} \delta u_2 \\ \delta\theta_3 \end{bmatrix}$$

# Deformation, stiffness, mass matrices and loads vector

- $\mathbf{B} = \nabla_u \mathbf{H}$ is the deformation matrix:

$$\mathbf{B} = \left[ \begin{array}{cc} \nabla_u h_1 & \nabla_u h_2 \end{array} \right] = \left[ \begin{array}{cccc} -1/\ell & x/\ell - 1 & 1/\ell & -x/\ell \\ 0 & -1/\ell & 0 & 1/\ell \end{array} \right]$$

- $\mathbf{K}$ and $\mathbf{M}$ and $\mathbf{r}$ are defined as

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega, \qquad \mathbf{M} = \int_\Omega \rho \, \mathbf{H}^T \mathbf{H} \, d\Omega, \qquad \mathbf{r}(t) = \int_{\Gamma_\sigma} \mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega \mathbf{H}^T \mathbf{f} \, d\Omega.$$