Linear elastodynamics

Solid elements

ME473 Dynamic finite element analysis of structures

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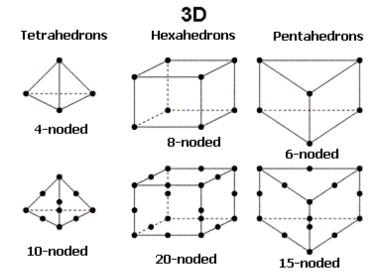
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation
3		FEM global	Project 1 statement
4		FEM local	Project 1
5		FEM local	Project 1 submission

Summary

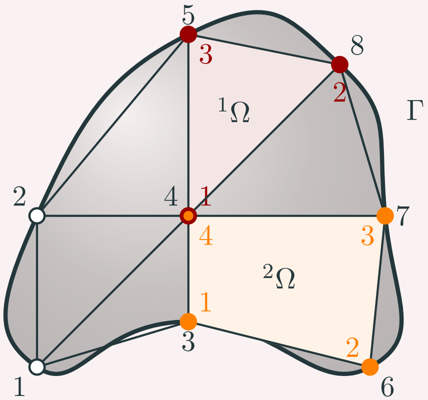
- Recap week 4
- 3D finite elements
- Isoparametric, subparametric, superparametric elements
- Numerical integration in 3d
- MATLAB PDE Toolbox Examples
- Abaqus example

Recommended readings

1. Gmür, Dynamique des structures (§3.3) [GM]
2. Neto et al., Engineering Computation of Structures (chap 7) [N]

Recap week 4

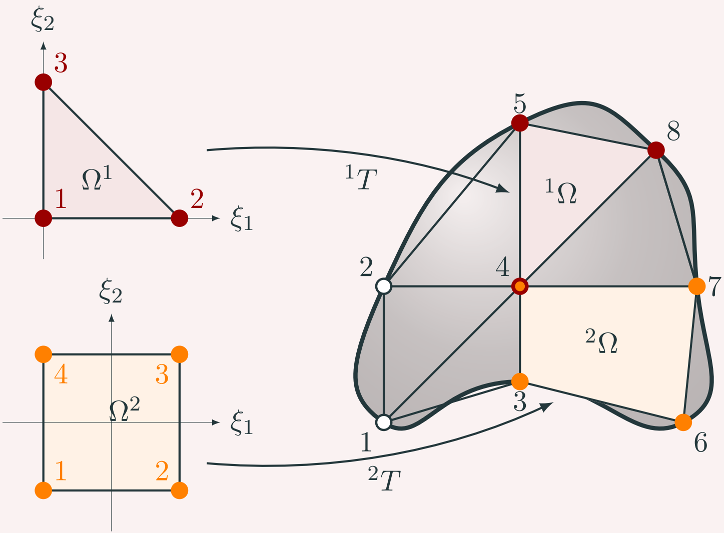
The local finite element point of view (in 2d)



Every finite element $^e\Omega$ has:

- Number of nodes $^e p$,
- Connectivity $(^e p \times 1)$,
- $^e\mathbf{q}$ vector of nodal displacements $(2\,^e p \times 1)$,
- $^e\mathbf{u}$ vector of displacements (2×1),
- $^e h_i$ local shape functions $(i = 1, \ldots, {}^e p)$,
- $^e\mathbf{H}$ local shape functions matrix $(2 \times 2\,^e p)$,
- $^e\mathbf{K}$ local stiffness matrix $(2\,^e p \times 2\,^e p)$,
- $^e\mathbf{M}$ local mass matrices $(2\,^e p \times 2\,^e p)$,
- $^e\mathbf{r}$ local loads vector $(2\,^e p \times 1)$.

Archetypal to local



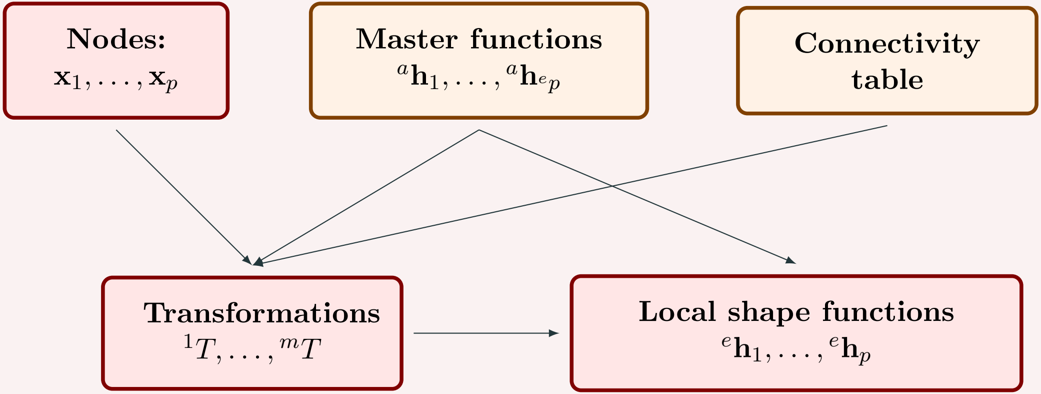
Paradigm shift: many with few

- ☞ **Main idea:** instead of working with local shape functions ${}^e h_i$ (one per node!) we will deduce them from a **small set of archetypal (or master) shape functions** ${}^a h_i$ defined on an archetypal element via a transformation ${}^e T$.
- ☞ **Clever idea:** the transformation ${}^e T$ is defined in terms of archetypal shape functions and nodes coordinates:

$$ {}^e T : \mathbf{x}(\boldsymbol{\xi}) = {}^a\mathbf{H}(\boldsymbol{\xi})\,{}^e\mathbf{x} = \sum_{i=1}^{{}^e p} {}^a h_i(\boldsymbol{\xi})\,{}^e\mathbf{x}_i $$

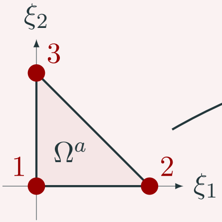
- ✗ Archetypal elements follows established conventions (node numbering and nodes coordinates) that one needs to adopt.

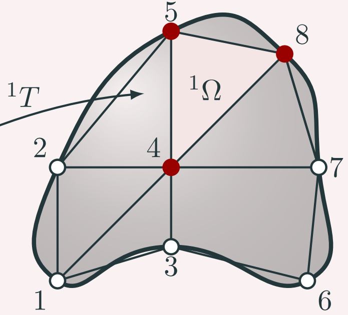
The clever idea



Concrete example: bilinear triangular element (2d)

Node id local	Local Coord	Node id global	Global coord.
1	$(0,0)$	4	(x_4, y_4)
2	$(1,0)$	8	(x_8, y_8)
3	$(0,1)$	5	(x_5, y_5)





Base (archetypal) functions:

$$^a h_1 = 1 - \xi_1 - \xi_2$$
$$^a h_2 = \xi_1$$
$$^a h_3 = \xi_2$$

Transformation:

$$^1T : \begin{cases} x(\xi_1, \xi_2) = x_4 h_1 + x_8 h_2 + x_5 h_3 = x_4 + (x_8 - x_4)\xi_1 + (x_5 - x_4)\xi_2 \\ y(\xi_1, \xi_2) = y_4 h_1 + y_8 h_2 + y_5 h_3 = y_4 + (y_8 - y_4)\xi_1 + (y_5 - y_4)\xi_2 \end{cases}$$

Consequence 1: local shape functions are useless

- The coordinate transformation ${}^eT^{-1}$ maps shape functions on ${}^e\Omega$ to the master element space Ω^a:
$$ {}^a\mathbf{H}(\boldsymbol{\xi}) = {}^e\mathbf{H}(\mathbf{x}(\boldsymbol{\xi})) $$
- An example of a relationship between local and archetypal shape function:
$$ {}^1h_8(x,y) = {}^ah_2\big(\xi_1(x,y), \xi_2(x,y)\big) $$

☞ In general, the inverse transformations ${}^eT^{-1} : \big(\xi_1(x,y), \xi_2(x,y)\big)$ are not easy to compute since $\mathbf{x}(\boldsymbol{\xi})$ are nonlinear.

☞ The local functions ${}^eh_j$ may have complex expressions, but they are not necessary to explicitly compute!

Consequence 2: integegration over archetypal elements

- Thanks to ${}^{e}T$, the integrals over ${}^{e}\Omega$ in the definitions of stiffness and mass matrices and loads vectors can be computed over the regular element Ω^a by performing a change of variables:

$$\int_{{}^{e}\Omega} (\dots) d\Omega \quad \xmapsto{\text{replaced by}} \quad \int_{\Omega^a} (\dots) {}^{e}j \, d\xi_1 d\xi_2$$

$$\frac{\partial}{\partial x_i} \quad \xmapsto{\text{replaced by}} \quad \frac{\partial \xi_1}{\partial x_i} \partial_{\xi_1} + \frac{\partial \xi_2}{\partial x_i} \partial_{\xi_2}$$

- For each transformation, it is necessary to compute the Jacobian matrix, its determinant, and its inverse.

Concrete example: bilinear triangular element (2d)

- **Transformation:**

$$
{}^1T : \begin{cases} x(\xi_1, \xi_2) = x_4 h_1 + x_8 h_2 + x_5 h_3 = x_4 + (x_8 - x_4)\xi_1 + (x_5 - x_4)\xi_2 \\ y(\xi_1, \xi_2) = y_4 h_1 + y_8 h_2 + y_5 h_3 = y_4 + (y_8 - y_4)\xi_1 + (y_5 - y_4)\xi_2 \end{cases}
$$

- **Jacobian matrix:**

$$
{}^1\mathbf{J} = \begin{bmatrix} \partial x/\partial \xi_1 & \partial x/\partial \xi_2 \\ \partial y/\partial \xi_1 & \partial y/\partial \xi_2 \end{bmatrix} = \begin{bmatrix} x_8 - x_4 & x_5 - x_4 \\ y_8 - y_4 & y_5 - y_4 \end{bmatrix}
$$

- **Jacobian determinant:**

$$
{}^1j = \det {}^1\mathbf{J} = (x_8 - x_4)(y_5 - y_4) - (x_5 - x_4)(y_8 - y_4)
$$

- **Jacobian inverse matrix:**

$$
{}^1\mathbf{J}^{-1} = \frac{1}{{}^1j} \begin{bmatrix} x_8 - x_4 & -x_5 + x_4 \\ -y_8 + y_4 & y_5 - y_4 \end{bmatrix}
$$

Concrete example: bilinear triangular element (2d)

- **Local mass matrix:**

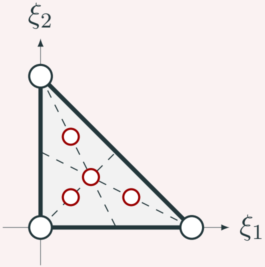
$$
\begin{aligned}
{}^{1}\mathbf{M} &= \int_{\Omega^a} \rho\, {}^{a}\mathbf{H}^{T}\, {}^{a}\mathbf{H}\, {}^{1}j \, d\boldsymbol{\xi} \\
&= \int_{\Omega^a} \rho \begin{bmatrix} {}^{a}h_1^2\mathbf{I} & {}^{a}h_1\,{}^{a}h_2\mathbf{I} & {}^{a}h_1\,{}^{a}h_3\mathbf{I} \\ {}^{a}h_2\,{}^{a}h_1\mathbf{I} & {}^{a}h_2^2\mathbf{I} & {}^{a}h_2\,{}^{a}h_3\mathbf{I} \\ {}^{a}h_3\,{}^{a}h_1\mathbf{I} & {}^{a}h_3\,{}^{a}h_2\mathbf{I} & {}^{a}h_3^2\mathbf{I} \end{bmatrix} {}^{1}j \, d\xi_2 d\xi_1
\end{aligned}
$$

- **Local stiffness matrix:**

$$
{}^{1}\mathbf{K} = \int_{\Omega^a} \begin{bmatrix} \cdots & & \cdots \\ & \left(\nabla_{\boldsymbol{\xi}}\, {}^{a}h_i\, {}^{1}\mathbf{J}^{-1}\right)^{T} \mathbf{C} \left(\nabla_{\boldsymbol{\xi}}\, {}^{a}h_j\, {}^{1}\mathbf{J}^{-1}\right) & \\ \cdots & & \cdots \end{bmatrix} {}^{1}j \, d\xi_2 d\xi_1
$$

Consequence 3: numerical integration: Gauss-Legendre

$$\int_{\Omega^a} f(\xi_1, \xi_2) d\xi_1 d\xi_2 \approx \sum_{i=1}^{4} \omega_i f(\xi_1^i, \xi_2^i)$$



Gauss points (ξ_1^i, ξ_2^i)	Weights (ω_i)
$(1/5, 1/5)$	$25/96$
$(3/5, 1/5)$	$25/96$
$(1/5, 3/5)$	$25/96$
$(1/3, 1/3)$	$9/32$

- If f is a polynomial of degree $d_i = 2r_i - 1$ in the variable ξ_i, then the Gauss-Legendre approximation formula is **exact**.

Assembly of the mass matrix: local to global

$$
{}^{1}\mathbf{M} = \int_{\Omega^a} \rho \begin{bmatrix} {}^a h_1^2 \mathbf{I} & {}^a h_1 {}^a h_2 \mathbf{I} & {}^a h_1 {}^a h_3 \mathbf{I} \\ {}^a h_2 {}^a h_1 \mathbf{I} & {}^a h_2^2 \mathbf{I} & {}^a h_2 {}^a h_3 \mathbf{I} \\ {}^a h_3 {}^a h_1 \mathbf{I} & {}^a h_3 {}^a h_2 \mathbf{I} & {}^a h_3^2 \mathbf{I} \end{bmatrix} {}^e j \, d\xi_2 d\xi_1 = \begin{bmatrix} {}^1\mathbf{M}_{11} & {}^1\mathbf{M}_{12} & {}^1\mathbf{M}_{13} \\ {}^1\mathbf{M}_{21} & {}^1\mathbf{M}_{22} & {}^1\mathbf{M}_{23} \\ {}^1\mathbf{M}_{31} & {}^1\mathbf{M}_{32} & {}^1\mathbf{M}_{33} \end{bmatrix}
$$

We assembly local mass matrices into the global ones using the assembly operator:

$$
\mathbf{M} = \overset{m}{\underset{e=1}{\mathsf{A}}} {}^e\mathbf{M} = \sum_{e=1}^{m} {}^e\mathbf{L}^{T\,e}\mathbf{M}^e\mathbf{L}
$$

where

$$
{}^1\mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

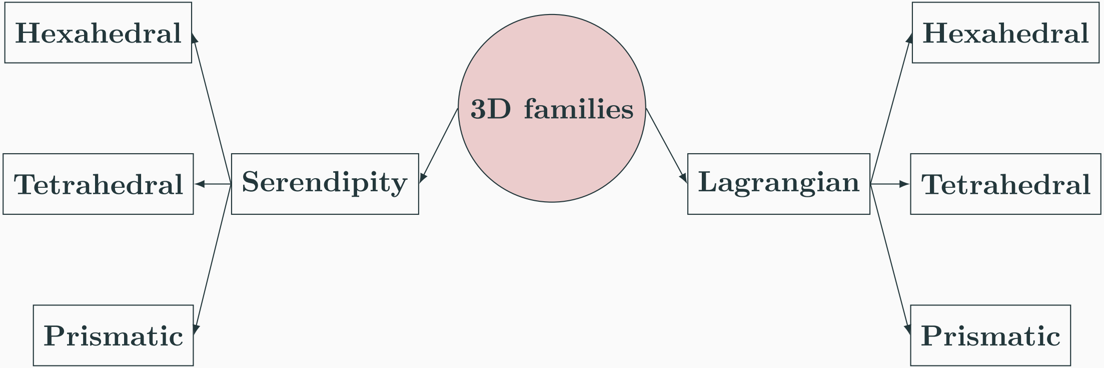
Node id. local	Node id. global
1	4
2	8
3	5

Assembly of the mass matrix: local to global

$$
{}^1\mathbf{L}^T\,{}^1\mathbf{M}\,{}^1\mathbf{L} = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & {}^1\mathbf{M}_{11} & {}^1\mathbf{M}_{13} & \mathbf{0} & \mathbf{0} & {}^1\mathbf{M}_{12} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & {}^1\mathbf{M}_{31} & {}^1\mathbf{M}_{33} & \mathbf{0} & \mathbf{0} & {}^1\mathbf{M}_{32} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & {}^1\mathbf{M}_{21} & {}^1\mathbf{M}_{23} & \mathbf{0} & \mathbf{0} & {}^1\mathbf{M}_{22}
\end{bmatrix}
$$

3D finite elements

Three-dimensional families of finite elements



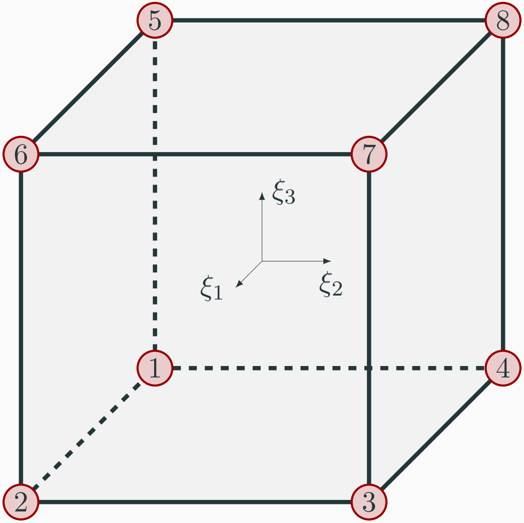
Serendipity vs Lagrangian elements

- **Lagrangian finite elements:**
  - Defined using Lagrange polynomials.
  - Nodes are placed at both element edges and inside the element.
  - Suitable for higher-order accuracy in both structured and unstructured meshes.
  - Used in applications where internal nodes improve interpolation.

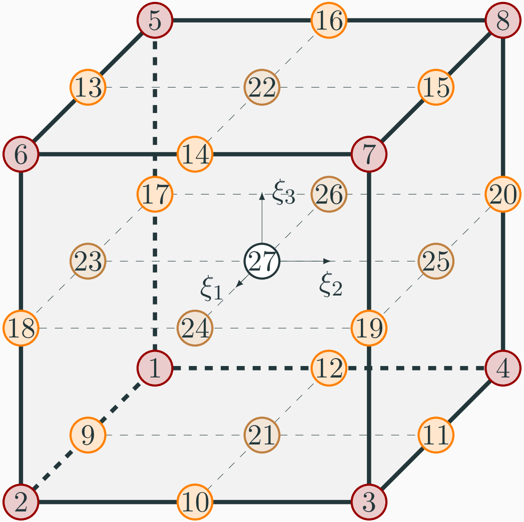
- **Serendipity finite elements:**
  - Defined using a reduced number of nodes compared to Lagrangian elements.
  - Nodes exist *only on the edges* (no interior nodes for 2D elements or nodes on faces for 3D elements).
  - More computationally efficient but less accurate for high-order approximations.
  - Preferred for quadrilateral and hexahedral elements in structured meshes to avoids certain types of instabilities.

Trilinear hexahedral (lagrangian) element

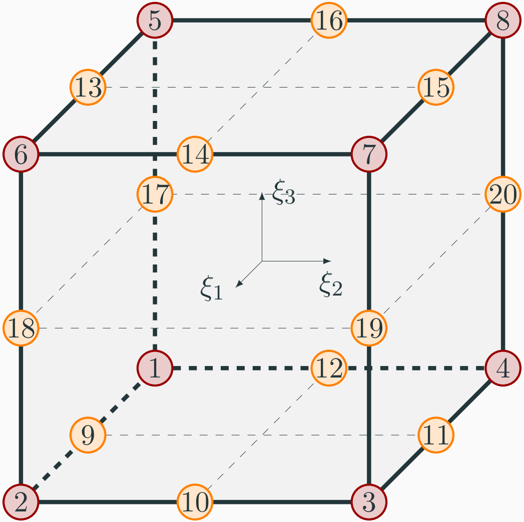
Node	Coordinates
1	$(-1, -1, -1)$
2	$(+1, -1, -1)$
3	$(+1, +1, -1)$
4	$(-1, +1, -1)$
5	$(-1, -1, +1)$
6	$(+1, -1, +1)$
7	$(+1, +1, +1)$
8	$(-1, +1, +1)$



Triquadratic hexahedral (lagrangian) element



Triquadratic hexahedral serendipity element



Trilinear hexahedral base functions

$$
\begin{aligned}
{}^a h_1 &= 0.125(1-\xi_1)(1-\xi_2)(1-\xi_3) \\
{}^a h_2 &= 0.125(1+\xi_1)(1-\xi_2)(1-\xi_3) \\
{}^a h_3 &= 0.125(1+\xi_1)(1+\xi_2)(1-\xi_3) \\
{}^a h_4 &= 0.125(1-\xi_1)(1+\xi_2)(1-\xi_3) \\
{}^a h_5 &= 0.125(1-\xi_1)(1-\xi_2)(1+\xi_3) \\
{}^a h_6 &= 0.125(1+\xi_1)(1-\xi_2)(1+\xi_3) \\
{}^a h_7 &= 0.125(1+\xi_1)(1+\xi_2)(1+\xi_3) \\
{}^a h_8 &= 0.125(1-\xi_1)(1+\xi_2)(1+\xi_3)
\end{aligned}
$$

Triquadratic hexahedral serendipity base functions

1. **Vertex nodes (1-8):**

$$ {}^a h_i = 0.125 \cdot (1 \pm \xi_1) \cdot (1 \pm \xi_2) \cdot (1 \pm \xi_3) $$

2. **Mid-edge nodes (9-20):**

$$ {}^a h_i = \begin{cases} 0.25 \cdot (1 - \xi_1^2) \cdot (1 \pm \xi_2) \cdot (1 \pm \xi_3) \\ 0.25 \cdot (1 \pm \xi_1) \cdot (1 - \xi_2^2) \cdot (1 \pm \xi_3) \\ 0.25 \cdot (1 \pm \xi_1) \cdot (1 \pm \xi_2) \cdot (1 - \xi_3^2) \end{cases} $$

3. Correction formula for vertex nodes (1-8):

$$ {}^a h_i \leftarrow {}^a h_i - 0.5 \cdot ({}^a h_{j_1} + {}^a h_{j_2} + {}^a h_{j_3}) $$

→ **Triquadratic hexahedral serendipity base functions**

Triquadratic hexahedral serendipity base functions

4. **Mid-face nodes (21-26):**

$$
{}^a h_i = \begin{cases} 0.5 \cdot (1 \pm \xi_1) \cdot (1 - \xi_2^2) \cdot (1 - \xi_3^2) \\ 0.5 \cdot (1 - \xi_1^2) \cdot (1 \pm \xi_2) \cdot (1 - \xi_3^2) \\ 0.5 \cdot (1 - \xi_1^2) \cdot (1 - \xi_2^2) \cdot (1 \pm \xi_3) \end{cases}
$$

5. Vertex nodes (1-8) adjustement:

$$
{}^a h_i \leftarrow {}^a h_i - 0.5 \cdot ({}^a h_{j_1} + {}^a h_{j_2} + {}^a h_{j_3})
$$

6. Mid-edge nodes (9-20) adjustement:

$$
{}^a h_i \leftarrow {}^a h_i - 0.5({}^a h_{j_1} + {}^a h_{j_2})
$$

Triquadratic hexahedral base functions

7. **Interior node (27):**

$$^a h_{27} = (1 - \xi_1^2)(1 - \xi_2^2)(1 - \xi_3^2)$$

8. Vertex nodes (1-8) adjustement:

$$^a h_i \leftarrow {}^a h_i - 0.125 {}^a h_{27}$$

9. Mid-edge nodes (9-20) adjustement:

$$^a h_i \leftarrow {}^a h_i + 0.25 {}^a h_{27}$$

10. Mid-face nodes (21-26) adjustement:

$$^a h_i \leftarrow {}^a h_i - 0.5 {}^a h_{27}$$

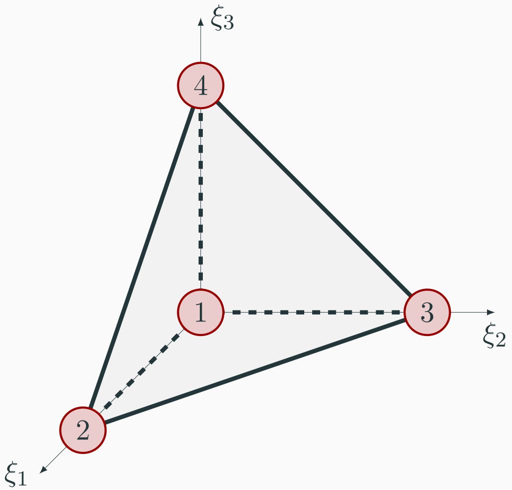
$\rightarrow$ **Triquadratic hexahedral base functions**

Trilinear tetrahedral element

Node	Coordinates
1	$(0,0,0)$
2	$(1,0,0)$
3	$(0,1,0)$
4	$(0,0,1)$

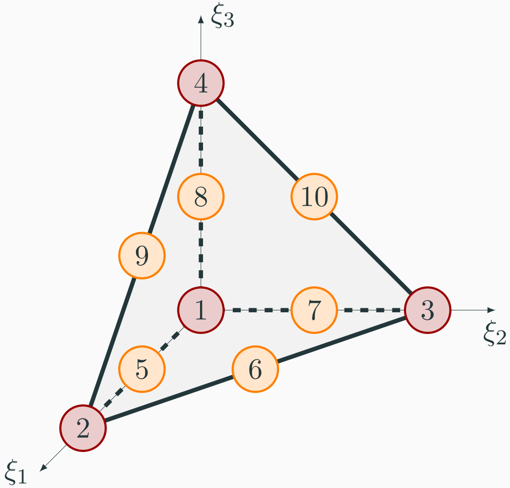
Base functions

$$
\begin{aligned}
{}^a h_1 &= 1-\xi_1-\xi_2-\xi_3 \\
{}^a h_2 &= \xi_1 \\
{}^a h_3 &= \xi_2 \\
{}^a h_4 &= \xi_3
\end{aligned}
$$



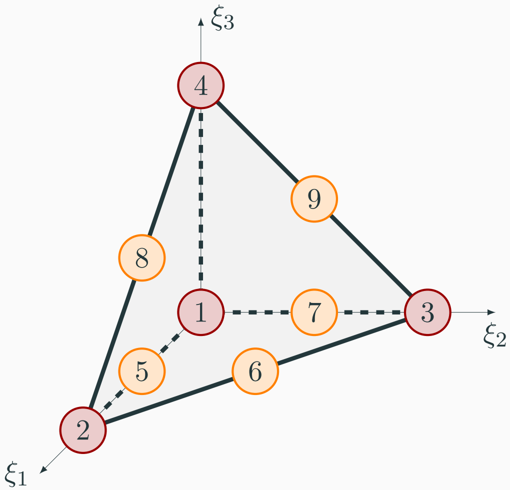
Triquadratic tetrahedral element

Node	Coordinates
1	$(0, 0, 0)$
2	$(1, 0, 0)$
3	$(0, 1, 0)$
4	$(0, 0, 1)$
5	$(0.5, 0, 0)$
6	$(0.5, 0.5, 0)$
7	$(0, 0.5, 0)$
8	$(0, 0, 0.5)$
9	$(0.5, 0, 0.5)$
10	$(0, 0.5, 0.5)$



Triquadratic tetrahedral serendipidy element

Node	Coordinates
1	$(0, 0, 0)$
2	$(1, 0, 0)$
3	$(0, 1, 0)$
4	$(0, 0, 1)$
5	$(0.5, 0, 0)$
6	$(0.5, 0.5, 0)$
7	$(0, 0.5, 0)$
8	$(0.5, 0, 0.5)$
9	$(0, 0.5, 0.5)$



Base functions for tetrahedral elements

Mid-edge nodes
$({}^a h_5, \ldots, {}^a h_{10})$

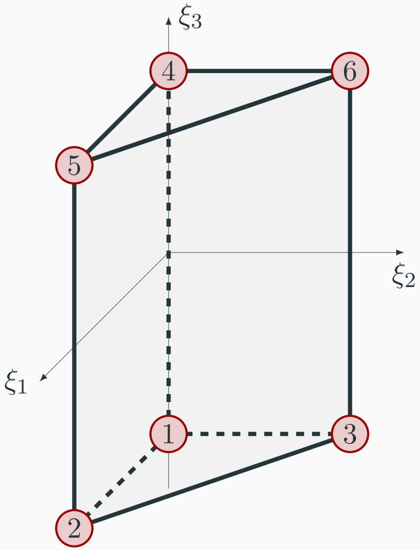
Trilinear tetrahedral base functions
Vertex nodes $({}^a h_1, \ldots, {}^a h_4)$

Triquadratic tetrahedral base functions
$({}^a h_1, \ldots, {}^a h_{10})$

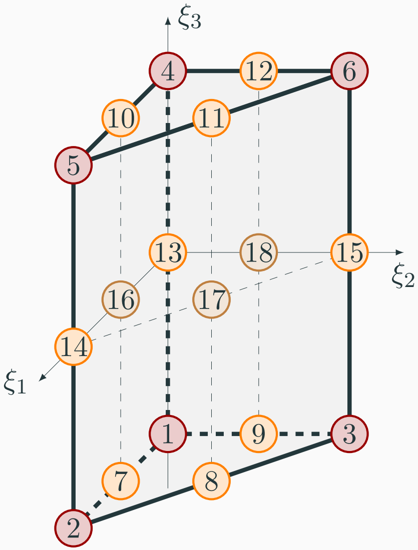
Correction for vertex nodes
$({}^a h_1, \ldots, {}^a h_4)$

Trilinear prismatic element

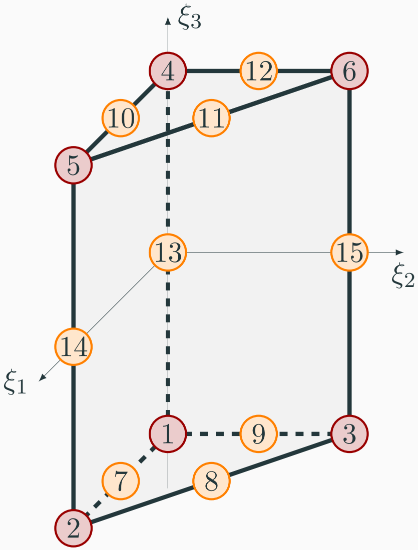
Node	Coordinates
1	$(0, 0, -1)$
2	$(1, 0, -1)$
3	$(0, 1, -1)$
4	$(0, 0, 1)$
5	$(1, 0, 1)$
6	$(0, 1, 1)$



Triquadratic prismatic element



Triquadratic prismatic serendipidy element



Base functions for prismatic elements

Trilinear prismatic element base functions

$$
\begin{aligned}
{}^a h_1 &= (1-\xi_1-\xi_2)(1-\xi_3)/2 & \qquad {}^a h_4 &= (1-\xi_1-\xi_2)(1+\xi_3)/2 \\
{}^a h_2 &= \xi_1(1-\xi_3)/2 & {}^a h_5 &= \xi_1(1+\xi_3)/2 \\
{}^a h_3 &= \xi_2(1-\xi_3)/2 & {}^a h_6 &= \xi_2(1+\xi_3)/2
\end{aligned}
$$

1. **Trilinear prismatic base functions**: corner nodes $({}^a h_1, \ldots, {}^a h_6)$
2. Mid-edge nodes definitions: $({}^a h_7, \ldots, {}^a h_{15})$
3. Correction for vertex nodes: $({}^a h_1, \ldots, {}^a h_6)$
4. **Triquadratic prismatic serendipidy base functions** $({}^a h_1, \ldots, {}^a h_{15})$
5. Mid-face nodes definitions: $({}^a h_{16}, {}^a h_{17}, {}^a h_{18})$
6. Correction for vertex nodes: $({}^a h_1, \ldots, {}^a h_6)$
7. Correction for mid-edge nodes: $({}^a h_7, \ldots, {}^a h_{15})$
8. **Triquadratic prismatic base functions** $({}^a h_1, \ldots, {}^a h_{18})$

Isoparametric, subparametric, superparametric elements

Geometrical representation and displacement approximation

Displacement field as well as the geometrical representation of the finite elements could be approximated using different sets of shape functions.

- **Geometrical representation**

$$^{e}T : \mathbf{x}(\boldsymbol{\xi}) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\,{}^{e}\mathbf{x}$$

- **Displacement field approximation**

$$^{e}\mathbf{u}^{h}(\mathbf{x}, t) = {}^{e}\mathbf{H}(\mathbf{x})\,{}^{e}\mathbf{q}(t)$$

1. Subparametric: less nodes for geometric than for displacement,
2. Isoparametric: same nodes for both geometry and displacement,
3. Superparametric: more nodes for geometric than for displacement.

Numerical integration in 3d

Integration over archetypal elements

$$\int_{\Omega^a} f \, d\Omega$$

Hexahedral

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f \, d\xi_1 d\xi_2 d\xi_3$$

Prismatic

$$\int_{0}^{1}\int_{0}^{1-\xi_1}\int_{-1}^{1} f \, d\xi_1 d\xi_2 d\xi_3$$

Tetrahedral

$$\int_{0}^{1}\int_{0}^{1-\xi_1}\int_{0}^{1-\xi_1-\xi_2} f \, d\xi_1 d\xi_2 d\xi_3$$

Gauss-Legendre numerical integration

Hexahedral

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f\, d\xi_1 d\xi_2 d\xi_3 \longrightarrow \approx \sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\sum_{k=1}^{r_3} \omega_i^1 \omega_j^2 \omega_k^3 f(\xi_1^i, \xi_2^j, \xi_3^k)$$

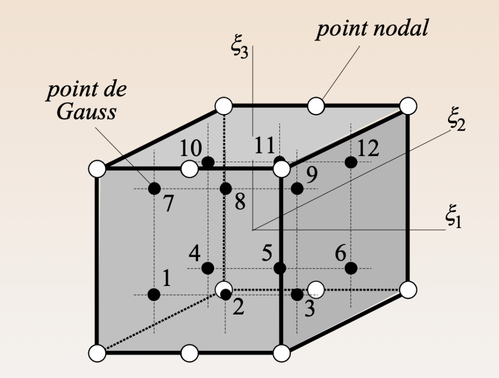
Prismatic

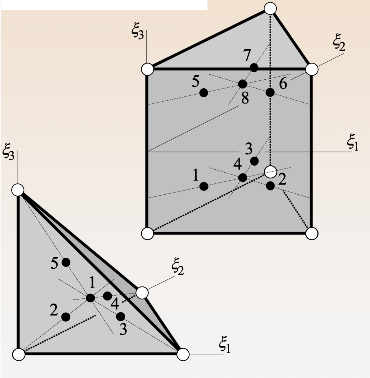
$$\int_{0}^{1}\int_{0}^{1-\xi_1}\int_{-1}^{1} f\, d\xi_1 d\xi_2 d\xi_3 \longrightarrow \approx \sum_{i=1}^{r_1}\sum_{k=1}^{r_3} \omega_i^{12} \omega_k^3 f(\xi_1^i, \xi_2^i, \xi_3^k)$$

Tetrahedral

$$\int_{0}^{1}\int_{0}^{1-\xi_1}\int_{0}^{1-\xi_1-\xi_2} f\, d\xi_1 d\xi_2 d\xi_3 \longrightarrow \approx \sum_{i=1}^{r_1} \omega_i^{123} f(\xi_1^i, \xi_2^i, \xi_3^i)$$

Examples of Gauss point distribution



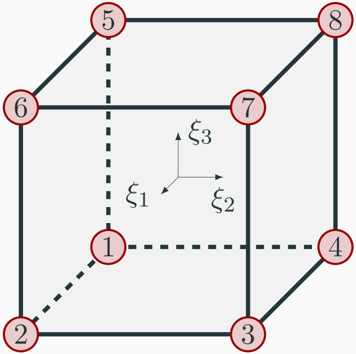


(Credit: Thomas Gmür - Dynamique numérique des solides et des structures)

Number of Gauss points for exact integration: trilinear hexahedral

- ${}^a h_i$ trilinear basis functions.
- Exact integration apparently with:
  - $2 \times 2 \times 2$ Gauss points for ${}^e\mathbf{M}$
  - $1 \times 1 \times 1$ Gauss points for ${}^e\mathbf{K}$
- Note that in case of distortion:
  - ${}^e j = {}^e j(\boldsymbol{\xi})$
  - $\frac{\partial {}^e\mathbf{H}}{\partial \mathbf{x}} = \frac{\partial {}^a\mathbf{H}}{\partial \xi} {}^e\mathbf{J}^{-1}$
  - ${}^e\mathbf{J}^{-1} = {}^e\mathbf{J}^{-1}(\boldsymbol{\xi})$ function of $\boldsymbol{\xi}$ at the denominator.

  Thus the Gauss quadrature will never be exact in case of distortion.



Number of Gauss points for exact integration

Rule for exact integration: evaluate the number of Gauss points on the archetypal element Ω^a (not on ${}^e\Omega$). Let n_i be the maximum number of nodes in direction ξ_i.

- **Hexahedral:** $n_1 \times n_2 \times n_3$ Gauss points.
- **Prismatic:** $n_{12} \times n_3$ Gauss points.
- **Tetrahedral:** n_{123} Gauss points

Example of exact integration for mass matrix:

- Trilinear Hexahedral: $n_1 \times n_2 \times n_3 = 2 \times 2 \times 2$
- Trilinear Tetrahedral: $n_{123} = 5$
- Trilinear Prismatic: $n_{12} \times n_3 = 4 \times 2$

Comparison of exact and reduced integration

Exact integration

- ✓ Integrals are computed exactly
- ✓ Monotonicity of convergence
- ✗ Overestimation of stiffness
- ✗ Risk of locking in linear elements

Reduced integration

- ✓ Reduction of computational costs
- ✗ Loss of monotonicity of convergence
- ✗ Hourglass effect in hexahedral structured meshes

Selective reduced integration:

- involves using fewer integration points in certain areas and more in others, based on the specific requirements of the analysis,
- reduce computational costs while maintaining accuracy in critical areas,
- mitigate the hourglass effect.