Linear elastodynamics

Finite element method in local coordinates

ME473 Dynamic finite element analysis of structures

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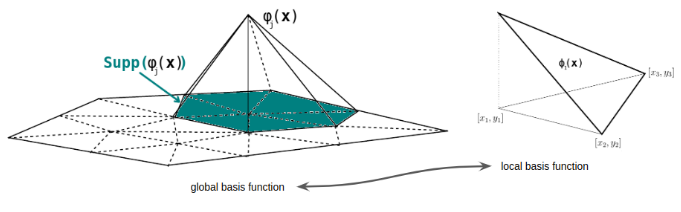
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation
3		FEM global	Project 1 statement
4		Solid 3D	Project 1

Summary

- Recap week 3
- Localization and elementary quantities
- Example: dynamic analysis of a clamped beam
- Automating integration and archetypal shape functions

Recommended readings

1. Gmür, Dynamique des structures (§3.3) [GM]
2. Neto et al., Engineering Computation of Structures (§2.3.5 - §2.3.8) [N]

Recap week 3

Displacements approximation in finte element method

Let p the number of nodes of the mesh.

$$\mathbf{u}^h(\mathbf{x},t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t) = \sum_{i=1}^{p} h_i(\mathbf{x})\mathbf{q}_i(t)$$

$$\delta\mathbf{u}^h(\mathbf{x}) = \mathbf{H}(\mathbf{x})\delta\mathbf{q} = \sum_{i=1}^{p} h_i(\mathbf{x})\boldsymbol{\delta}\mathbf{q}_i$$

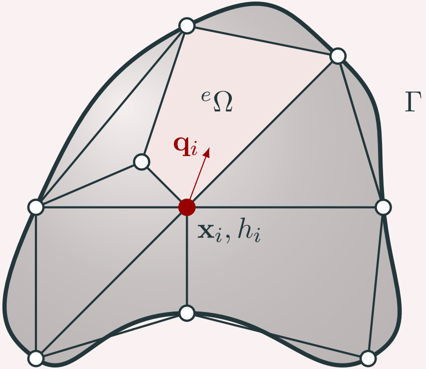
- $\mathbf{H}(\mathbf{x})$ is a $3 \times 3p$ matrix of **shape functions**:

$$\mathbf{H} = \left[\begin{array}{c:c:c:c:c:c} h_1\mathbf{I} & h_2\mathbf{I} & \dots & h_i\mathbf{I} & \dots & h_p\mathbf{I} \end{array} \right]$$

  $\mathbf{I}$ is the 3×3 identity matrix.

- $\mathbf{q}(t)$ is a $3p \times 1$ vector of (*unknown*) nodal displacements.
- $\delta\mathbf{q}$ is a $3p \times 1$ vector of virtual nodal displacements.

Global nodal shape functions requirements



Propretries of h_i:

- Linearly independent polynomial basis.
- Satisfy Kronecker delta property:

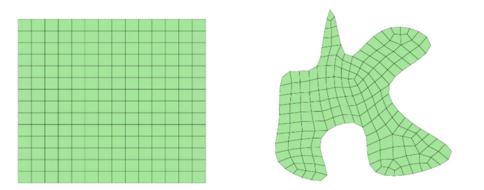
$$h_i(\mathbf{x}_i) = 1 \quad \text{and} \quad h_i(\mathbf{x}_j) = 0.$$

- Vanish on non-adjacent elements.
- Continuous at interfaces.
- Differentiable inside elements.
- Ensure rigid body motion & constant deformations.

Drawbacks of the global approach

- ✗ Limited capability in handling complex (unstructured) mesh topologies.
- ✗ Computationally expensive: it requires defining one shape function per node.
- ✗ Limited utilization of the compact support of nodal shape functions.

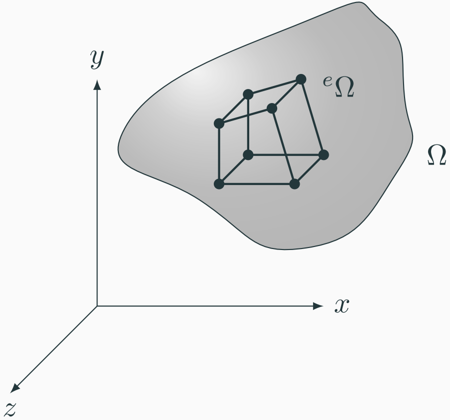
Local approach: provides a quicker and more sistematic way to compute the stiffenss and mass matrices and the applied forces vector:



(Credit: Onscale - structured vs unstructured meshes)

Localization and elementary quantities

Localization



- Let p be the number of nodes in the mesh.
- Let m be the number of finite elements in the mesh.
- Let $^e\Omega$ a finite element in the mesh.
- Let ^ep the number of nodes in the element $^e\Omega$.

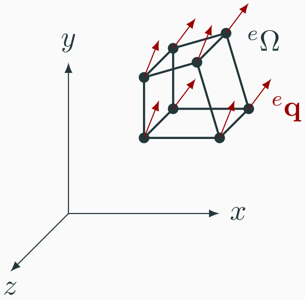
Localization of displacements

Restriction of displacements $\mathbf{u}^h$ and $\delta\mathbf{u}^h$ on the finite element ${}^e\Omega$:

$$ {}^e\mathbf{u}^h(\mathbf{x},t) = {}^e\mathbf{H}(\mathbf{x})\,{}^e\mathbf{q}(t) \qquad {}^e\boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}) = {}^e\mathbf{H}(\mathbf{x})\,{}^e\boldsymbol{\delta}\mathbf{q} $$

- ${}^e\mathbf{u}^h$ *restriction* (3×1) of the displacement vector $\mathbf{u}^h$ on the finite element ${}^e\Omega$.
- ${}^e\boldsymbol{\delta}\mathbf{u}^h$ *restriction* (3×1) of the virtual dispolecement vector $\boldsymbol{\delta}\mathbf{u}^h$ on the finite element ${}^e\Omega$.
- ${}^e\mathbf{H}$ matrix $(3 \times 3^e p)$ of elementary shape functions of the finite element ${}^e\Omega$.
- ${}^e\mathbf{q}$ vector $(3^e p \times 1)$ of unknown nodal displacements in the finite element ${}^e\Omega$.
- ${}^e\boldsymbol{\delta}\mathbf{q}$ vector $(3^e p \times 1)$ of nodal displacements in the finite element ${}^e\Omega$.

Local displacements approximation



$$^e\mathbf{u}^h = \left[\ {}^eh_1\mathbf{I} \;\vdots\; {}^eh_2\mathbf{I} \;\vdots\; \dots \;\vdots\; {}^eh_{{}^ep}\mathbf{I}\ \right] \begin{pmatrix} {}^e\mathbf{q}_1 \\ {}^e\mathbf{q}_2 \\ \vdots \\ {}^e\mathbf{q}_{{}^ep} \end{pmatrix}$$

$$^e\boldsymbol{\delta}\mathbf{u}^h = \left[\ {}^eh_1\mathbf{I} \;\vdots\; {}^eh_2\mathbf{I} \;\vdots\; \dots \;\vdots\; {}^eh_{{}^ep}\mathbf{I}\ \right] \begin{pmatrix} {}^e\boldsymbol{\delta}\mathbf{q}_1 \\ {}^e\boldsymbol{\delta}\mathbf{q}_2 \\ \vdots \\ {}^e\boldsymbol{\delta}\mathbf{q}_{{}^ep} \end{pmatrix}$$

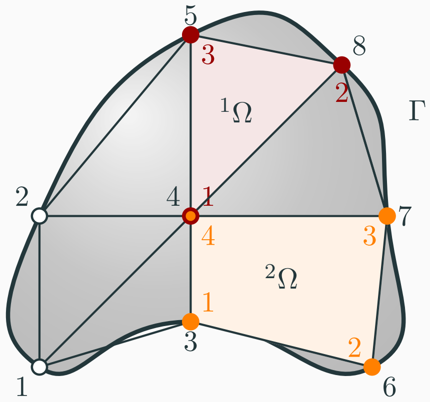
$$
\underbrace{\begin{bmatrix} {}^e\mathbf{q}_1 \\ \vdots \\ {}^e\mathbf{q}_{{}^e p} \end{bmatrix}}_{3^e p \times 1}
=
\underbrace{\begin{bmatrix} {}^e\mathbf{L}_{11} & \dots & {}^e\mathbf{L}_{1p} \\ \vdots & & \vdots \\ {}^e\mathbf{L}_{{}^e p1} & \dots & {}^e\mathbf{L}_{{}^e pp} \end{bmatrix}}_{3^e p}
\cdot
\underbrace{\begin{bmatrix} \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_p \end{bmatrix}}_{3p \times 1}
$$

$$ {}^e\mathbf{q} = {}^e\mathbf{L}\mathbf{q} $$

${}^e\mathbf{L}$ is a Boolean location matrix:

- ${}^e\mathbf{L}_{ij} = \mathbf{I}$ (3×3 identity matrix) if global node j corresponds to local node i,
- ${}^e\mathbf{L}_{ij} = \mathbf{0}$ (3×3 null matrix) otherwise.

Localization matrices - example



- Number of nodes in the mesh: $p = 8$.
- Number of elements in the mesh: $m = 6$.
- Number of nodes in the element $^1\Omega$: $^1p = 3$.

$$^1\mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

- Number of nodes in the element $^2\Omega$: $^2p = 4$.

$$^2\mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Practical example:

Consider a mesh made of

- $p = 10'000$ nodes,
- trilinear hexahedral finite elements with ${}^e p = 8$ nodes each.

Every localization matrix ${}^e\mathbf{L}$ contains

- $720'000$ entries,
- of which $3\,{}^e p = 24$ are 1s,

✗ the remaining $719'976$ are 0s.

Connectivity table: local to global node numbering

- Elements and their connectivity are defined using a table.
- Example connectivity table:

${}^e\Omega$	${}^1\Omega$	${}^2\Omega$	${}^3\Omega$	${}^4\Omega$
1	1	2	4	5
2	2	3	5	6
3	5	6	8	9
4	4	5	7	8

- The connectivity matrix provides the global numbering for each node in each element, corresponding to a column in the table above.
- The localization matrix ${}^e\mathbf{L}$ is constructed from the connectivity table.

Additivity of integrals

- Approximated weak form:

$$\int_{\Omega} (\nabla \boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{C} \nabla \mathbf{u}^h \, d\Omega + \int_{\Omega} \rho (\boldsymbol{\delta}\mathbf{u}^h)^T \ddot{\mathbf{u}}^h \, d\Omega = \int_{\Gamma_\sigma} (\boldsymbol{\delta}\mathbf{u}^h)^T \hat{\mathbf{f}} \, d\Gamma + \int_{\Omega} (\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{f} \, d\Omega$$

- We localize the integration using the additivity of integrals:

$$\sum_{e=1}^{m} \left(\int_{{}^e\Omega} (\nabla {}^e\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{C} \nabla {}^e\mathbf{u}^h \, d\Omega + \int_{{}^e\Omega} \rho ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T {}^e\ddot{\mathbf{u}}^h \, d\Omega \right)$$
$$= \sum_{e=1}^{m} \left(\int_{{}^e\Gamma_\sigma} ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T \hat{\mathbf{f}} \, d\Gamma + \int_{{}^e\Omega} ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{f} \, d\Omega \right)$$

and consider the local quantities ${}^e\mathbf{u}^h = {}^e\mathbf{H} {}^e\mathbf{L}\mathbf{q}$ and ${}^e\boldsymbol{\delta}\mathbf{u}^h = {}^e\mathbf{H} {}^e\mathbf{L}\boldsymbol{\delta}\mathbf{q}$.

Additivity of integrals - inertial forces

- Recall ${}^e\mathbf{u}^h = {}^e\mathbf{H}{}^e\mathbf{L}\mathbf{q}$ and ${}^e\boldsymbol{\delta}\mathbf{u}^h = {}^e\mathbf{H}{}^e\mathbf{L}\boldsymbol{\delta}\mathbf{q}$.
- Consider only the term related to the virtual work of inertial forces (acceleration):

$$\sum_{e=1}^{m} \int_{{}^e\Omega} \rho ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T {}^e\ddot{\mathbf{u}}^h \, d\Omega = \boldsymbol{\delta}\mathbf{q}^T \Big[\sum_{e=1}^{m} {}^e\mathbf{L}^T \underbrace{\Big(\int_{{}^e\Omega} \rho {}^e\mathbf{H}^T {}^e\mathbf{H} \, d\Omega \Big)}_{{}^e\mathbf{M}} {}^e\mathbf{L} \Big] \ddot{\mathbf{q}}$$

Additivity of integrals - internal and external forces

- Recall ${}^e\mathbf{u}^h = {}^e\mathbf{H}{}^e\mathbf{L}\mathbf{q}$ and ${}^e\boldsymbol{\delta}\mathbf{u}^h = {}^e\mathbf{H}{}^e\mathbf{L}\boldsymbol{\delta}\mathbf{q}$.
- Analogously for the term related to the virtual work of internal forces:

$$\sum_{e=1}^{m} \int_{{}^e\Omega} (\nabla {}^e\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{C} \nabla {}^e\mathbf{u}^h \, d\Omega = \boldsymbol{\delta}\mathbf{q}^T \Big[\sum_{e=1}^{m} {}^e\mathbf{L}^T \underbrace{\Big(\int_{{}^e\Omega} \nabla {}^e\mathbf{H}^T \mathbf{C} \nabla {}^e\mathbf{H} \, d\Omega \Big)}_{{}^e\mathbf{K}} {}^e\mathbf{L} \Big] \mathbf{q}$$

- Term realted to the virtual work of external forces:

$$\sum_{e=1}^{m} \int_{{}^e\Gamma_\sigma} ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T \hat{\mathbf{f}} \, d\Gamma + \int_{{}^e\Omega} ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{f} \, d\Omega$$

$$= \boldsymbol{\delta}\mathbf{q}^T \sum_{e=1}^{m} {}^e\mathbf{L}^T \underbrace{\Big(\int_{{}^e\Gamma_\sigma} {}^e\mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_{{}^e\Omega} {}^e\mathbf{H}^T \mathbf{f} \, d\Omega \Big)}_{{}^e\mathbf{r}(t)}$$

Elementary matrices and vectors

- **Elementary stiffness matrix** $(3^e p \times 3^e p)$:

$$^e\mathbf{K} = \int_{^e\Omega} {^e\mathbf{B}^T} \mathbf{C} {^e\mathbf{B}} \, d\Omega$$

$$^e\mathbf{B} = \nabla^e\mathbf{H} = \left[\begin{array}{c:c:c} \nabla^e h_1 & \dots & \nabla^e h_{^e p} \end{array} \right] \text{ elementery deformation matrix } (6 \times 3^e p) \; .$$

- **Elementary mass matrix** $(3^e p \times 3^e p)$:

$$^e\mathbf{M} = \int_{^e\Omega} \rho \, {^e\mathbf{H}^T} {^e\mathbf{H}} \, d\Omega.$$

- **Elementary applied forces vector** $(3^e p \times 1)$:

$$^e\mathbf{r}(t) = \int_{^e\Gamma_\sigma} {^e\mathbf{H}^T} \hat{\mathbf{f}} \, d\Gamma + \int_{^e\Omega} {^e\mathbf{H}^T} \mathbf{f} \, d\Omega.$$

Assembly operator

We define the assembly operator as follows:

$$\mathbf{K} = \mathop{\mathrm{A}}_{e=1}^{m} {}^{e}\mathbf{K} = \sum_{e=1}^{p} {}^{e}\mathbf{L}^{T\,e}\mathbf{K}^{e}\mathbf{L}$$

$$\mathbf{M} = \mathop{\mathrm{A}}_{e=1}^{m} {}^{e}\mathbf{M} = \sum_{e=1}^{p} {}^{e}\mathbf{L}^{T\,e}\mathbf{M}^{e}\mathbf{L}$$

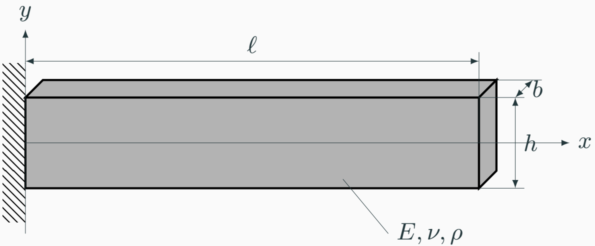
$$\mathbf{r} = \mathop{\mathrm{A}}_{e=1}^{m} {}^{e}\mathbf{r} = \sum_{e=1}^{p} {}^{e}\mathbf{L}^{T\,e}\mathbf{r}$$

Example: modal analysis of a clamped beam

Modal analysis of a clamped beam

Kinematic assumptions:

- The beam is made of elastic material which is homogeneous and isotropic (E, ν and ρ).
- Assume a plane stress state (very small thickness). The structure can be modeled in the (x, y) plane using two-dimensional finite elements.



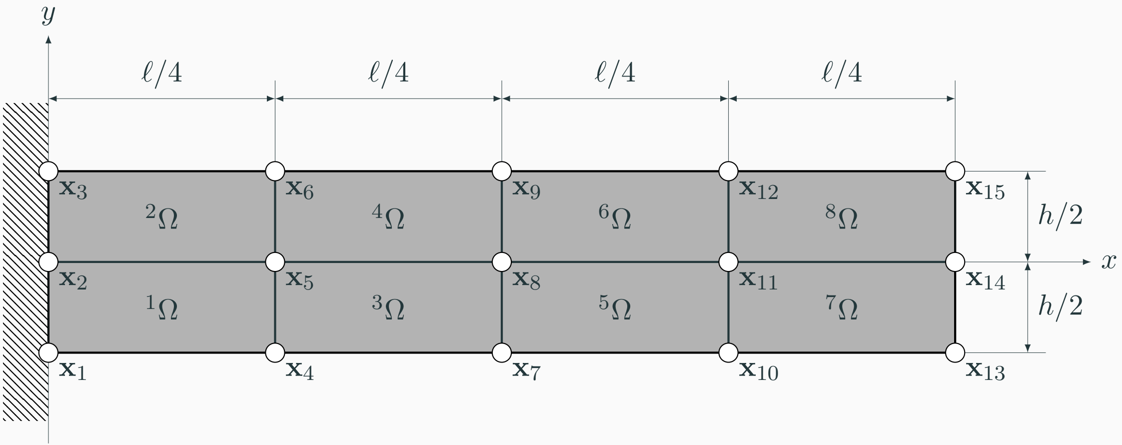
- E Young's modulus
- ν Poisson's ratio
- ρ material density
- ℓ length
- h heigth
- b thickness
- x axial coordinate
- y transversal coordinate

Variables:

- $u_1(x, y, t)$ axial displacement
- $u_2(x, y, t)$ transversal displacement

Modal analysis of a clamped beam

Discretization into 8 bilinear quadrilateral finite elements (4 nodes each)



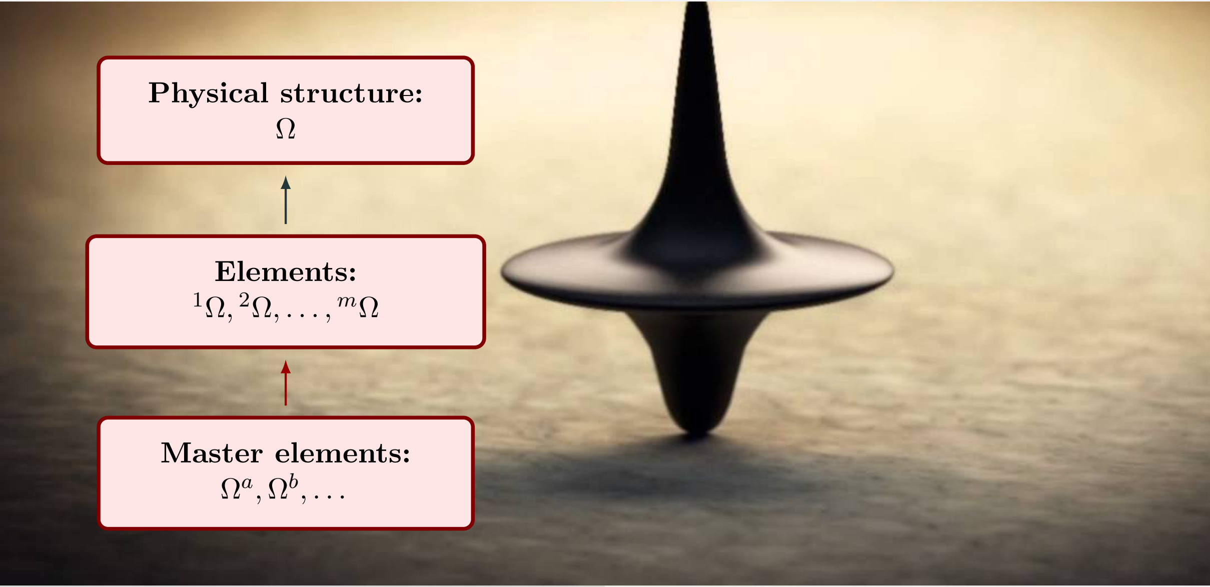
Modal analysis of a clamped beam

Objective: determine the first natural frequencies of the beam.

▸ Go to Matlab Drive

Automating integration and archetypal shape functions

Elements



Physical structure:
Ω

↑

Elements:
${}^1\Omega, {}^2\Omega, \ldots, {}^m\Omega$

↑

Master elements:
$\Omega^a, \Omega^b, \ldots$

Coordinate transform

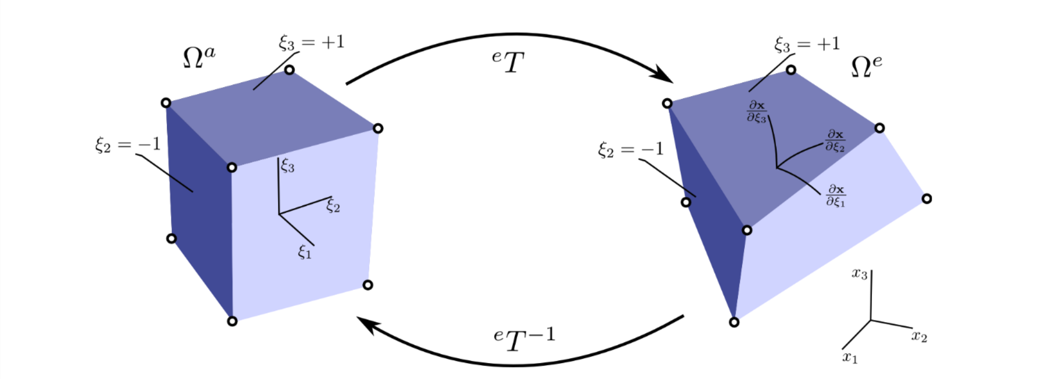
To automate the integration and simplify the definition of shape functions, we transform each *distorted* elementary domain ${}^e\Omega$ into a **reference (archetypal or master) domain** Ω^a where we can apply standard numerical integration schemes.

- The coordinate transformation:

$$\begin{aligned} {}^eT : \Omega^a &\to {}^e\Omega \\ \boldsymbol{\xi} &\mapsto \mathbf{x}(\boldsymbol{\xi}) \end{aligned}$$

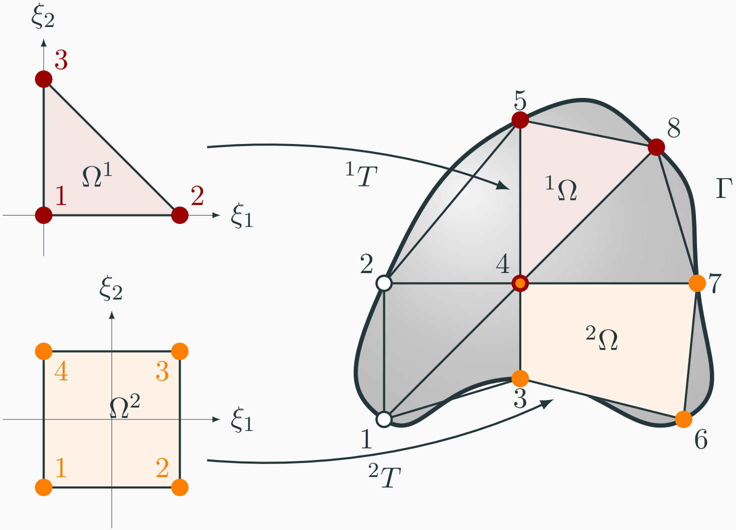
  maps any point $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}^T$ in Ω^a to its corresponding point of coordinate $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) = \{x_1(\boldsymbol{\xi}), x_2(\boldsymbol{\xi}), x_3(\boldsymbol{\xi})\}^T$ in ${}^e\Omega$.
- ${}^eT$ is a bijective application.

An illustration of a coordinate transform ${}^eT$ in 3D



(Credit: Joel Cugnoni - Finite Element Method applied to linear statics of deformable solids)

An illustration of coordinate transforms in 2D



Biunivocity of coordinate transformation ${}^eT$

$$
{}^e\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_2} \\ \frac{\partial x_1}{\partial \xi_3} & \frac{\partial x_2}{\partial \xi_3} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix} \qquad {}^e\mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_3}{\partial x_1} \\ \frac{\partial \xi_1}{\partial x_2} & \frac{\partial \xi_2}{\partial x_2} & \frac{\partial \xi_3}{\partial x_2} \\ \frac{\partial \xi_1}{\partial x_3} & \frac{\partial \xi_2}{\partial x_3} & \frac{\partial \xi_3}{\partial x_3} \end{bmatrix}
$$

- ${}^e\mathbf{J}$ is the Jacobian matrix associated with ${}^eT$: ${}^e\mathbf{J}_{ij} = \frac{\partial x_i}{\partial \xi_j}$ $(i, j = 1, 2, 3)$,
- ${}^e\mathbf{J}^{-1}$ is the inverse Jacobian matrix,
- ${}^ej = \det({}^e\mathbf{J})$ is the determinant of the Jacobian matrix ${}^e\mathbf{J}$.

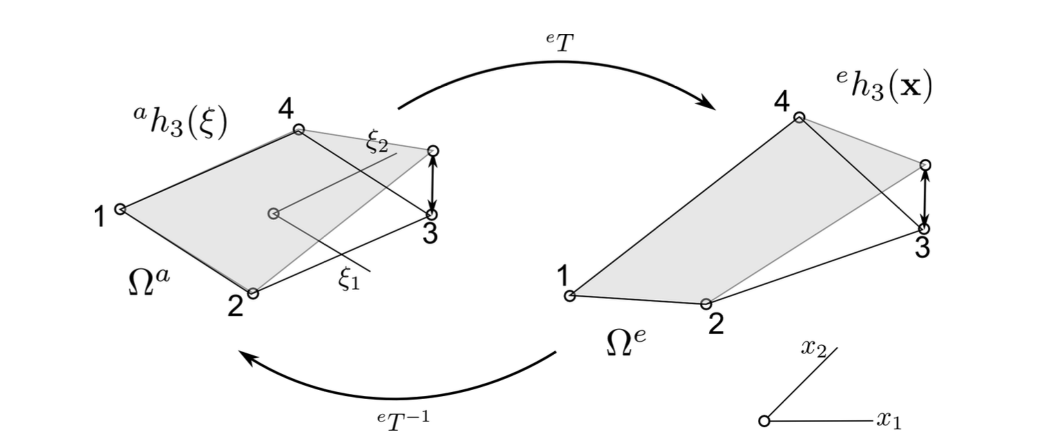
Sufficient condition for invertibility: if ${}^ej > 0$ everywhere in ${}^e\Omega$, then ${}^eT$ is invertible in ${}^e\Omega$ and ${}^e\mathbf{J}^{-1}$ exists.

Master elements and master shape functions

- The coordinate transformation ${}^eT^{-1}$ maps shape functions on ${}^e\Omega$ to the master element space Ω^a:
$$ {}^a\mathbf{H}(\boldsymbol{\xi}) = {}^e\mathbf{H}(\mathbf{x}(\boldsymbol{\xi})) $$
- Inside each element ${}^e\Omega$:
$$ {}^e\mathbf{u}^h[\mathbf{x}(\boldsymbol{\xi})] = {}^a\mathbf{H}(\boldsymbol{\xi})\,{}^e\mathbf{q} $$
- This allows shape function ${}^a\mathbf{H}(\boldsymbol{\xi})$ to be defined only *once* on each master element ${}^a\Omega$.

☞ **Result: master shape functions!**

An illustration of a master shape function



(Credit: Joel Cugnoni - Finite Element Method applied to linear statics of deformable solids)

Choosing a simple coordinate transformation

☞ Coordinate transformations are defined using the shape functions:

$$ {}^{e}T : \mathbf{x}(\boldsymbol{\xi}) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\,{}^{e}\mathbf{x} = \sum_{i=1}^{{}^{e}p} {}^{a}h_i(\boldsymbol{\xi})\,{}^{e}\mathbf{x}_i $$

- Inside each element ${}^{e}\Omega$, the local coordinates are interpolated as a linear combination of master shape functions ${}^{a}h_i$ and nodal coordinates ${}^{e}\mathbf{x}_i$.
- Kronecker property ensures node correspondence:

$$ {}^{e}h_i(\mathbf{x}_j) = \delta_{ij} \quad \Rightarrow \quad {}^{a}h_i(\boldsymbol{\xi}_j) = \delta_{ij}. $$

- This guarantees that each node of the master element Ω^{a} maps to a corresponding node in the deformed element ${}^{e}\Omega$.

Integration by substitution formulas

- Given ${}^eT : \Omega^a \to {}^e\Omega$, an integral over ${}^e\Omega$ of a function $F : {}^e\Omega \to \mathbb{R}$ can be rewritten as an integral over Ω^a:

$$\int_{{}^e\Omega} F(\mathbf{x})\, d\mathbf{x} = \int_{\Omega^a} F(\mathbf{x}(\boldsymbol{\xi}))\, {}^e j\, d\boldsymbol{\xi}$$

- When the integrand involves the operator ∇, then:

$$\int_{{}^e\Omega} \nabla_{\mathbf{x}} F(\mathbf{x})\, d\mathbf{x} = \int_{\Omega^a} \nabla_{\boldsymbol{\xi}} F(\mathbf{x}(\boldsymbol{\xi}))^e\mathbf{J}^{-1}\, {}^e j\, d\boldsymbol{\xi}$$

since

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j} = \sum_{j=1}^{3} {}^e\mathbf{J}_{ij}^{-1} \frac{\partial}{\partial \xi_j}.$$

Master elements and derivatives

- The spatial derivative operator $\nabla_{\mathbf{x}}$ is defined in the global coordinate system (x_1, x_2, x_3). Applying the coordinate transform ${}^{e}T^{-1}$, we can then extend it to be applied on the master element Ω^{a}:

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j} = \sum_{j=1}^{3} {}^{e}\mathbf{J}^{-1}_{ij} \frac{\partial}{\partial \xi_j}.$$

- The elementary strain-displacement matrix ${}^{e}\mathbf{B}$ can be directly derived from the master shape functions ${}^{a}\mathbf{H}$:

$${}^{e}\mathbf{B} = \left[\begin{array}{c:c:c} \nabla_{\mathbf{x}} {}^{e}h_1 & \dots & \nabla_{\mathbf{x}} {}^{e}h_{e_p} \end{array}\right] = \left[\begin{array}{c:c:c} \nabla_{\boldsymbol{\xi}} {}^{a}h_1 {}^{e}\mathbf{J}^{-1} & \dots & \nabla_{\boldsymbol{\xi}} {}^{a}h_{e_p} {}^{e}\mathbf{J}^{-1} \end{array}\right].$$

Using the coordinate transform ${}^{e}T$ and master shape functions, the integrals in the definitions of ${}^{e}\mathbf{K}$, ${}^{e}\mathbf{M}$ and ${}^{e}\mathbf{r}$, can be carried out directly on a standard domain Ω^{a}:

$$
{}^{e}\mathbf{K} = \int_{\Omega^{a}} \left(\nabla_{\boldsymbol{\xi}}\,{}^{a}\mathbf{H}\,{}^{e}\mathbf{J}^{-1}\right)^{T} \mathbf{C} \left({}^{e}\nabla_{\boldsymbol{\xi}}\,{}^{a}\mathbf{H}\,{}^{e}\mathbf{J}^{-1}\right)\, {}^{e}j \, d\boldsymbol{\xi}
$$

$$
{}^{e}\mathbf{M} = \int_{\Omega^{a}} \rho\, {}^{a}\mathbf{H}^{T}\,{}^{a}\mathbf{H}\, {}^{e}j \, d\boldsymbol{\xi}
$$

$$
{}^{e}\mathbf{r}(t) = \int_{\Gamma_{\sigma}^{a}} {}^{a}\mathbf{H}^{T}\hat{\mathbf{f}}\, {}^{e}j\big|_{\Gamma_{\sigma}^{a}} \, d\Gamma + \int_{\Omega^{a}} {}^{a}\mathbf{H}^{T}\mathbf{f}\, {}^{e}j \, d\boldsymbol{\xi}
$$

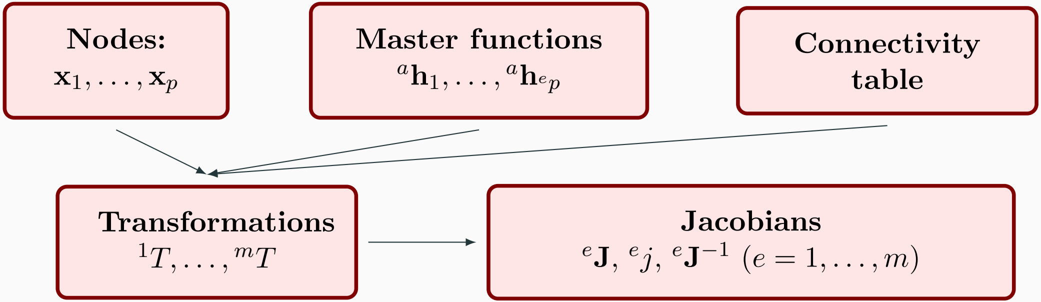
Systematization of the algorithm

Nodes
$\mathbf{x}_1, \ldots, \mathbf{x}_p$

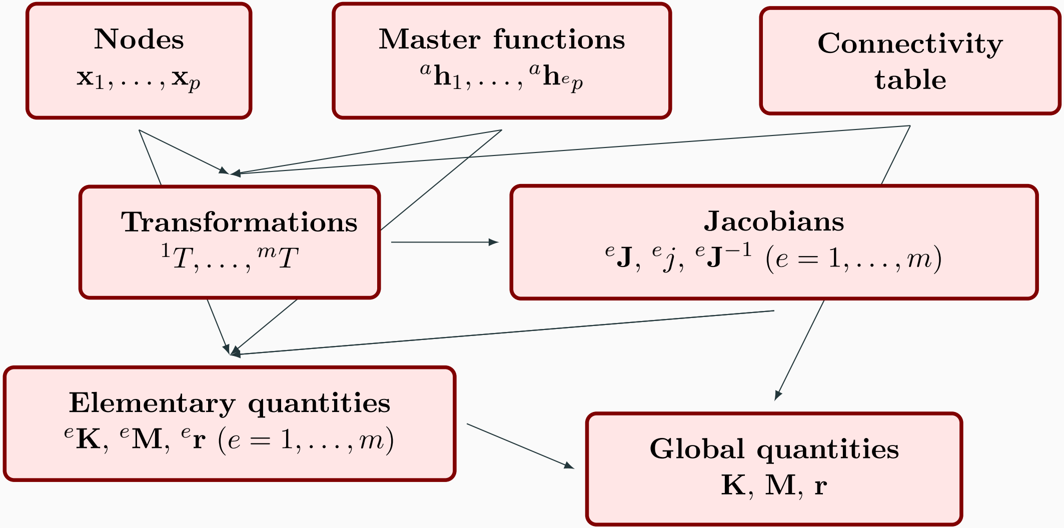
Master functions
${}^a\mathbf{h}_1, \ldots, {}^a\mathbf{h}_{{}^e p}$

Connectivity table

Systematization of the algorithm



Systematization of the algorithm



Numerical integration: Gauss-Legendre

- Numerical integration helps automate the calculation of the elementary quantities ${}^e\mathbf{K}$, ${}^e\mathbf{M}$ and ${}^e\mathbf{r}$.
- Integration is approximated as:

$$\int_{\Omega^a} f(\boldsymbol{\xi}) d\boldsymbol{\xi} \approx \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} \sum_{k=1}^{r_3} \omega_i^1 \omega_j^2 \omega_k^3 f(\xi_1^i, \xi_2^j, \xi_3^k)$$

  where ξ_j^i are known as *Gauss points* and ω_i^j are their associated *Gauss weights*.
- If f is a polynomial of degree $d_i = 2r_i - 1$ in the variable ξ_i, then the Gauss-Legendre approximation formula is **exact**.

Numerical integration in practice

This allows us to compute the stiffness matrix, the mass matrix and the load vector numerically as:

$$
{}^{e}\mathbf{K} \approx \sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\sum_{k=1}^{r_3} \omega_i^1 \omega_j^2 \omega_k^3 \left[(\nabla_{\boldsymbol{\xi}} {}^{a}\mathbf{H} {}^{e}\mathbf{J}^{-1})^T \mathbf{C} (\nabla_{\boldsymbol{\xi}} {}^{a}\mathbf{H} {}^{e}\mathbf{J}^{-1}) {}^{e}j \right]_{\xi_1=\xi_1^i, \xi_2=\xi_2^j, \xi_3=\xi_3^k}
$$

$$
{}^{e}\mathbf{M} \approx \sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\sum_{k=1}^{r_3} \omega_i^1 \omega_j^2 \omega_k^3 \left[\rho \, {}^{a}\mathbf{H}^T \, {}^{a}\mathbf{H} \right]_{\xi_1=\xi_1^i, \xi_2=\xi_2^j, \xi_3=\xi_3^k}
$$

$$
{}^{e}\mathbf{r} \approx \sum_{i=1}^{r_1}\sum_{j=1}^{r_2} \tilde{\omega}_i^1 \tilde{\omega}_j^2 \left[{}^{a}\mathbf{H}^T \hat{\mathbf{f}} \right]_{\xi_1=\tilde{\xi}_1^i, \xi_2=\tilde{\xi}_2^j} + \sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\sum_{k=1}^{r_3} \omega_i^1 \omega_j^2 \omega_k^3 \left[{}^{a}\mathbf{H}^T \mathbf{f} \right]_{\xi_1=\xi_1^i, \xi_2=\xi_2^j, \xi_3=\xi_3^k}
$$

An illustration of Gauss points