1D models

- Displacements: $\mathbf{u} = \{u_1\}$
- Strain: $\boldsymbol{\varepsilon} = \{\varepsilon_{11}\}$
- Stress: $\boldsymbol{\sigma} = \{\sigma_{11}\}$
- Linear strain-displacement relationship: $\varepsilon_{11} = \partial_{x_1} u_1$

$$\nabla = \left[\partial_{x_1}\right]$$

- Generalised Hooke's law: $\sigma_{11} = E\varepsilon_{11}$
- Stiffness material matrix:

$$\mathbf{C} = \left[E\right]$$

- Dynamic equilibrium equation:

$$\partial_{x_1}\sigma_{11} + f_1 = \rho\ddot{u}_1 \qquad \text{or} \qquad E\partial^2_{x_1x_1}u_1 + f_1 = \rho\ddot{u}_1$$

2D models - plane stress

- Displacements: $\mathbf{u} = \{u_1, u_2\}^T$
- Strain: $\boldsymbol{\varepsilon} = \{\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}\}^T$, $\varepsilon_{23} = \varepsilon_{13} = 0$ and $\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$
- Stress: $\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$, $\sigma_{11} = \sigma_{23} = \sigma_{13} = 0$
- Linear strain-displacement relationship: $\boldsymbol{\varepsilon} = \nabla\mathbf{u}$

$$\nabla = \begin{bmatrix} \partial_{x_1} & 0 \\ 0 & \partial_{x_2} \\ \partial_{x_2} & \partial_{x_1} \end{bmatrix}$$

- Generalised Hooke's law: $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$
- Stiffness isotropic material matrix:

$$\mathbf{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

- Dynamic equilibrium equations:

$$\begin{cases} \partial_{x_1}\sigma_{11} + \partial_{x_2}\sigma_{12} + f_1 = \rho\ddot{u}_1 \\ \partial_{x_2}\sigma_{22} + \partial_{x_1}\sigma_{12} + f_2 = \rho\ddot{u}_2 \end{cases} \qquad \text{or} \qquad \nabla^T\mathbf{C}\nabla\mathbf{u} + \mathbf{f} = \rho\ddot{\mathbf{u}}$$

2D models - plane strain

- Displacements: $\mathbf{u} = \{u_1, u_2\}^T$, $u_3 = 0$.
- Strain: $\boldsymbol{\varepsilon} = \{\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}\}^T$, $\varepsilon_{33} = \varepsilon_{23} = \varepsilon_{13} = 0$
- Stress: $\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$, $\sigma_{23} = \sigma_{13} = 0$ and $\sigma_{33} = \frac{E\nu(\varepsilon_{11}+\varepsilon_{22})}{(1+\nu)(1+2\nu)}$
- Linear strain-displacement relationship: $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$

$$\nabla = \begin{bmatrix} \partial_{x_1} & 0 \\ 0 & \partial_{x_2} \\ \partial_{x_2} & \partial_{x_1} \end{bmatrix}$$

- Generalised Hooke's law: $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$
- Stiffness isotropic material matrix:

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

- Dynamic equilibrium equations:

$$\begin{cases} \partial_{x_1}\sigma_{11} + \partial_{x_2}\sigma_{12} + f_1 = \rho\ddot{u}_1 \\ \partial_{x_2}\sigma_{22} + \partial_{x_1}\sigma_{12} + f_2 = \rho\ddot{u}_2 \end{cases} \qquad \text{or} \qquad \nabla^T\mathbf{C}\nabla\mathbf{u} + \mathbf{f} = \rho\ddot{\mathbf{u}}$$

3D models

- Displacements: $\mathbf{u} = \{u_1, u_2, u_3\}^T$
- Strain: $\boldsymbol{\varepsilon} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}\}^T$
- Stress: $\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, 2\sigma_{23}, 2\sigma_{31}, 2\sigma_{12}\}^T$
- Linear strain-displacement relationship: $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$

$$\nabla = \begin{bmatrix} \partial_{x_1} & 0 & 0 \\ 0 & \partial_{x_2} & 0 \\ 0 & 0 & \partial_{x_3} \\ 0 & \partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & \partial_{x_1} \\ \partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix}$$

- Generalised Hooke's law: $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$

- Stiffness isotropic material matrix:

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

- Dynamic equilibrium equation:

$$\begin{cases} \partial_{x_1}\sigma_{11} + \partial_{x_2}\sigma_{12} + \partial_{x_3}\sigma_{13} + f_1 = \rho\ddot{u}_1 \\ \partial_{x_1}\sigma_{12} + \partial_{x_2}\sigma_{22} + \partial_{x_3}\sigma_{23} + f_2 = \rho\ddot{u}_2 \\ \partial_{x_1}\sigma_{13} + \partial_{x_2}\sigma_{23} + \partial_{x_3}\sigma_{33} + f_3 = \rho\ddot{u}_3 \end{cases} \qquad \text{or} \qquad \nabla^T\mathbf{C}\nabla\mathbf{u} + \mathbf{f} = \rho\ddot{\mathbf{u}}$$