Linear elastodynamics

Finite element method in global coordinates

ME473 Dynamic finite element analysis of structures

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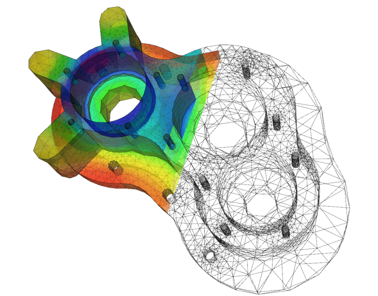
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation
3		FEM global	Project 1 statement

Summary

- Recap week 2
- Historical evolution of finite element method
- Finite element method : general ideas
- Finite element method in global coordinate system
- 1D and 2D elements and shape functions
- Example: longitudinal vibration of a bar

Recommended readings

1. Gmür, Dynamique des structures (§3.1 and §3.2) [GM]
2. Gmür, Méthode des éléments finis (§ 5.3) [GM1]
3. Neto et al., Engineering Computation of Structures (§ 2.3.1 - § 2.3.4) [N]

Recap week 2

Formulations of elastodynamics

Strong form:

$$\nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

↓

Weak form:

$$\int_\Omega (\nabla \boldsymbol{\delta}\mathbf{u})^T \mathbf{C} \nabla \mathbf{u} \, d\Omega + \int_\Omega \rho \, \boldsymbol{\delta}\mathbf{u}^T \ddot{\mathbf{u}} \, d\Omega = \int_{\Gamma_\sigma} \boldsymbol{\delta}\mathbf{u}^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega \boldsymbol{\delta}\mathbf{u}^T \mathbf{f} \, d\Omega$$

↓

Semi-discrete weak form:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t)$$

Galerkin method

- **Shape functions:** let $\mathbf{u}^h(\mathbf{x}, t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t)$ and $\delta\mathbf{u}^h(\mathbf{x}) = \mathbf{H}(\mathbf{x})\delta\mathbf{q}$.
- **Semi-discrete weak form of elastodynamics** find $\mathbf{q} \in C^2([0, T], \mathbb{R}^n)$ such that for every $\delta\mathbf{q} \in \mathbb{R}^n$ we have

$$\delta\mathbf{q}^T\left[\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) - \mathbf{r}(t)\right] = 0$$

coupled with initial conditions:

$$\delta\mathbf{q}^T\left(\mathbf{q}(0) - \mathbf{q_0}\right) = 0,$$

$$\delta\mathbf{q}^T\left(\dot{\mathbf{q}}(0) - \mathbf{p_0}\right) = 0.$$

Definitions

- **Stiffness matrix** $(n \times n)$:

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega$$

where $\mathbf{B}$ is the $(6 \times n)$ deformation matrix defined by $\mathbf{B} = \nabla \mathbf{H}$.

- **Mass matrix** $(n \times n)$:

$$\mathbf{M} = \int_{\Omega} \rho \, \mathbf{H}^T \mathbf{H} \, d\Omega.$$

- **Applied forces vector** $(n \times 1)$:

$$\mathbf{r}(t) = \int_{\Gamma_\sigma} \mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_{\Omega} \mathbf{H}^T \mathbf{f} \, d\Omega.$$

Advantages and drawbacks of Galerkin method

Advantages:

- ✓ Converges quickly with appropriate shape functions.
- ✓ Provides a systematic and structured approach for approximating solutions.
- ✓ The same set of functions is used to express real and virtual variables.

Drawbacks:

- ✗ Accuracy heavily dependent on choice of basis functions.
- ✗ No physical interpretation of the unknown variable $\mathbf{q}(t)$.
- ✗ The formulation of initial and boundary conditions in the discretized form is cumbersome.

Static, modal, and transient analysis

- **Static analysis**: determines deformations due to constant loads.

$$\mathbf{K}\mathbf{q} = \mathbf{r}$$

- **Modal analysis**: studies dynamic properties in the frequency domain.

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}$$

  - Natural frequency: key structural property essential in structural design.
    - Lower frequencies → higher displacement amplitudes → more dangerous.
    - Resonance occurs when external excitation frequency matches a natural frequency.
    - Prolonged resonance → catastrophic failure.

- **Transient analysis**: examines time-dependent structural responses to time-dependent excitations.

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t)$$

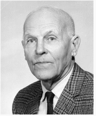
Historical evolution of finite element method

Theoretical formulation of finite element method

The FEM is the confluence of three ingredients: matrix structural analysis, the variational approach and a computer.
(source: Carlos A. Felippa)

1908 "*Lösung von Variationsproblemen*" by W. Ritz
1915 "*Weak formulation*" by B. Galerkin
1943 "*Mathematical foundation*" by R. Courant & A. Hrennikoff

Continuum discretization



1943

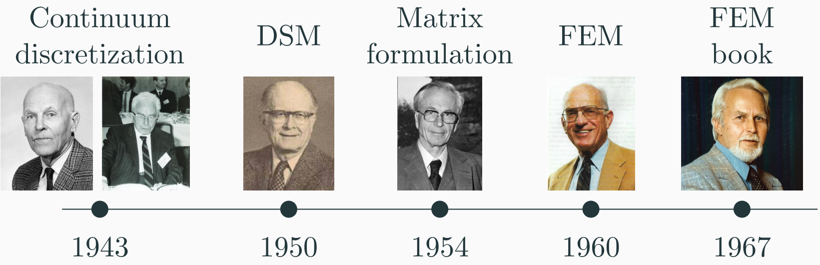
The birth of the finite element method

1950 Direct Stiffness Method (DSM) by M.J. Turner

1954 Matrix formulation of structural analysis by Agyris

1960 The term “Finite Element” was coined by Clough

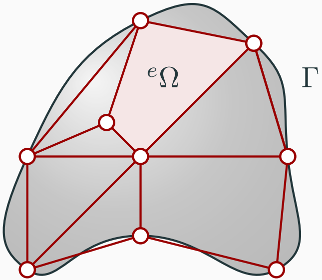
1967 First book on FEM by Zienkiewicz, Taylor, and Zhu



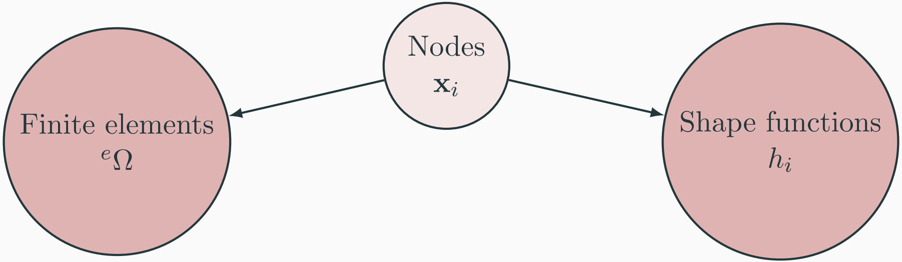
Finite element method : general ideas

Divide and conquer mentality

- The geometry of a structure is discretised when it is split into a mesh of finite elements ${}^1\Omega, \ldots, {}^m\Omega$ (smaller components or regular shapes).
- The discretisation introduces another approximation: this discretisation error can be reduced by using a finer mesh (i.e. more elements), or by increasing the accuracy of the finite elements chosen.
- Discretisation starts with the definition of a **set of nodes**: $\mathbf{x}_i$, $i = 1, \ldots, p$.



Nodes, elements and shape functions



- **Finite element method:** a special case of the Galerkin method, where shape functions are systematically defined in terms of nodes.
- Each shape function is non-zero only on a very limited number of finite elements (compact support).
- The computation of the mass and stiffness matrix in enhanced by the local nature of the shape functions.

Decompose the structure into its fundamental components

Idealization → **Discretization** → **Solution**

Example: the forth bridge. (source: A critical analysis of the forth bridge)

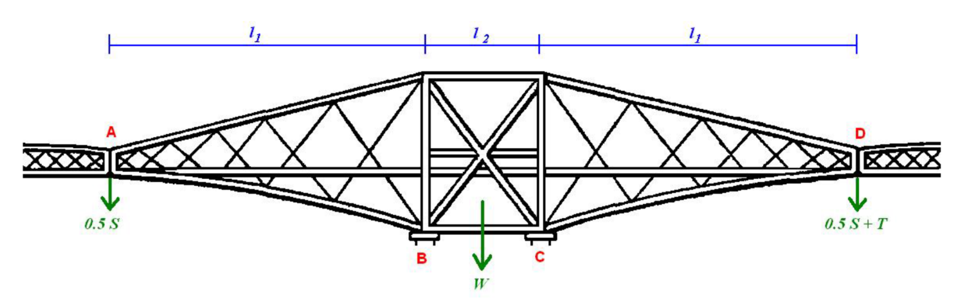
(Credit: theforthbridges.org)

Develop a simplified idealisation of the structure, which can often be accomplished through conceptual abstraction.

Decompose the structure into its fundamental components

Idealization → Discretization → Solution

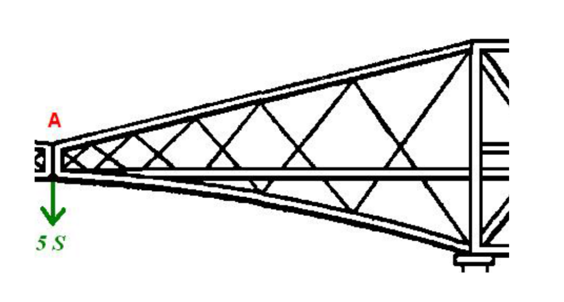
Conceptual model:



(Credit: Chatzi and Egger)

Idealization → Discretization → Solution

Physical system:

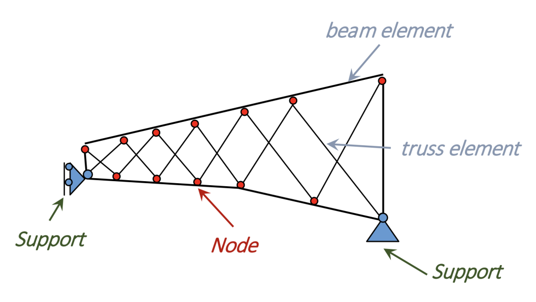


(Credit: Chatzi and Egger)

Various structural models can be formulated with differing levels of complexity in both geometric and mathematical representations.

Decompose the structure into its fundamental components

Mathematical model:

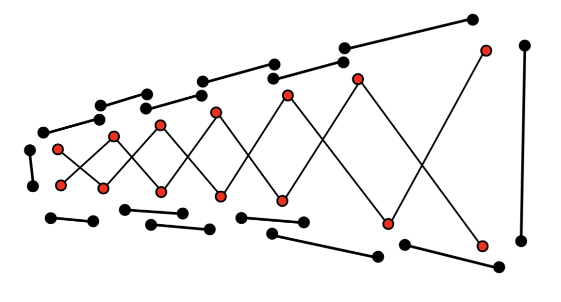


(Credit: Chatzi and Egger)

Assumption: decompose the system into 1D components, such as trusses and beams, to facilitate analysis.

Decompose the structure into its fundamental components

Direct stiffness method framework:

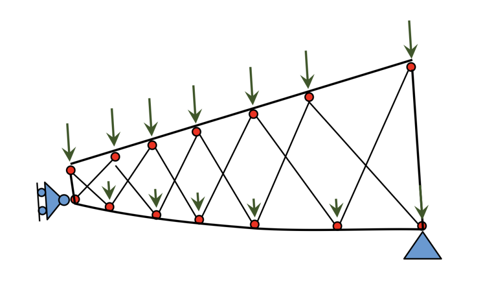


The continuum is disassembled using a mesh of finite elements that are connected at nodes located on the element boundaries.

(Credit: Chatzi and Egger)

Idealization → Discretization → **Solution**

Solve for degrees of freedom: displacements/rotations:



(Credit: Chatzi and Egger)

The solution process involves:

- **globalisation:** assembly of element equation and application of boundary conditions,
- **post-processing:** computation of strains, stresses, forces, and moments.

Advantages and drawbacks of finite element method

Advantages:

- ✓ Versatile for various engineering problems: mechanics of solids and fluids, dynamics, heat transfer, electrostatics, and more.
- ✓ Provides a systematic algorithm: defines the underlying shape functions used in the approximation.
- ✓ Accuracy control: solution precision can be improved by refining the mesh.

Drawbacks:

- ✗ Choice of mathematical model is crucial: the solution's reliability hinges on selecting an appropriate model.
- ✗ Computational cost: highly refined models increase the computational burden.
- ✗ Requires experience in meshing, boundary conditions, and solver settings.

Finite element method in global coordinate system

Let p the number of nodes of the mesh.

$$\mathbf{u}^h(\mathbf{x},t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t)$$

$$\delta\mathbf{u}^h(\mathbf{x}) = \mathbf{H}(\mathbf{x})\delta\mathbf{q}$$

- $\mathbf{H}(\mathbf{x})$ is a $3 \times 3p$ matrix of **shape functions**.
- $\mathbf{q}(t)$ is a $3p \times 1$ vector of (*unknown*) time-dependent functions.
- $\delta\mathbf{q}$ is a $3p \times 1$ vector of constants.

$$\mathbf{H} = \left[\begin{array}{c:c:c:c:c:c} h_1\mathbf{I} & h_2\mathbf{I} & \ldots & h_i\mathbf{I} & \ldots & h_p\mathbf{I} \end{array}\right]$$

- $\mathbf{I}$ is the 3×3 identity matrix.

Displacements approximation - matrix notation

$$\begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \end{pmatrix} = \left[\begin{array}{ccc:ccc:c:ccc} h_1 & 0 & 0 & h_2 & 0 & 0 & & h_p & 0 & 0 \\ 0 & h_1 & 0 & 0 & h_2 & 0 & \dots & 0 & h_p & 0 \\ 0 & 0 & h_1 & 0 & 0 & h_2 & & 0 & 0 & h_p \end{array} \right] \begin{pmatrix} q_{1,1} \\ q_{1,2} \\ q_{1,3} \\ \vdots \\ q_{p,1} \\ q_{p,2} \\ q_{p,3} \end{pmatrix}$$

$$\begin{pmatrix} \delta u_1^h \\ \delta u_2^h \\ \delta u_3^h \end{pmatrix} = \left[\begin{array}{ccc:ccc:c:ccc} h_1 & 0 & 0 & h_2 & 0 & 0 & & h_p & 0 & 0 \\ 0 & h_1 & 0 & 0 & h_2 & 0 & \dots & 0 & h_p & 0 \\ 0 & 0 & h_1 & 0 & 0 & h_2 & & 0 & 0 & h_p \end{array} \right] \begin{pmatrix} \delta q_{1,1} \\ \delta q_{1,2} \\ \delta q_{1,3} \\ \vdots \\ \delta q_{p,1} \\ \delta q_{p,2} \\ \delta q_{p,3} \end{pmatrix}$$

Displacements approximation - index notation

$$\mathbf{u}^h(\mathbf{x}, t) = \sum_{i=1}^{p} h_i(\mathbf{x})\mathbf{q}_i(t)$$

$$\boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}, t) = \sum_{i=1}^{p} h_i(\mathbf{x})\boldsymbol{\delta}\mathbf{q}_i$$

$$\mathbf{q}_i(t) = \begin{pmatrix} q_{i,1}(t) \\ q_{i,2}(t) \\ q_{i,3}(t) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\delta}\mathbf{q}^i = \begin{pmatrix} \delta q_{i,1} \\ \delta q_{i,2} \\ \delta q_{i,3} \end{pmatrix}$$

Deformation, stiffness, mass matrices and loads vector

- $\mathbf{B} = \nabla \mathbf{H}$ is a $(6 \times 3p)$ matrix:

$$\mathbf{B} = \left[\begin{array}{ccc:c:ccc} \partial_x h_1 & 0 & 0 & & \partial_x h_p & 0 & 0 \\ 0 & \partial_y h_1 & 0 & \dots & 0 & \partial_y h_p & 0 \\ 0 & 0 & \partial_z h_1 & & 0 & 0 & \partial_z h_p \\ 0 & \partial_z h_1 & \partial_y h_1 & & 0 & \partial_z h_p & \partial_y h_p \\ \partial_z h_1 & 0 & \partial_x h_1 & \dots & \partial_z h_p & 0 & \partial_x h_p \\ \partial_y h_1 & \partial_x h_1 & 0 & & \partial_y h_p & \partial_x h_p & 0 \end{array}\right] = \left[\begin{array}{c:c:c} \nabla h_1 & \dots & \nabla h_p \end{array}\right]$$

- $\mathbf{K}$ and $\mathbf{M}$ are $(3p \times 3p)$ matrices and $\mathbf{r}$ is a $(3p \times 1)$ vector:

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega, \qquad \mathbf{M} = \int_\Omega \rho \, \mathbf{H}^T \mathbf{H} \, d\Omega, \qquad \mathbf{r}(t) = \int_{\Gamma_\sigma} \mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega \mathbf{H}^T \mathbf{f} \, d\Omega.$$

Global nodal shape functions - construction guidelines

Global nodal shape functions $h_i : \Omega \to \mathbb{R}$ are characterized by the following properties:

- They form a linearly independent basis of polynomials of a given degree.
- Their values lie in the interval $[0, 1]$.
- They satisfy the Kronecker delta property: $h_i(\mathbf{x}_i) = 1$ and $h_i(\mathbf{x}_j) = 0$ for $i \neq j$.
- They vanish on all finite elements that are not adjacent to the node $\mathbf{x}_i$.

Properties of shape functions - convergence criteria

- Continuity of h_i at nodes and interfaces of finite elements.
- Differentiability of h_i inside each finite element.
- Completeness (rigid body motion and constant deformation):

$$\sum_{i=1}^{p} h_i(\mathbf{x}) = 1 \quad \text{and} \quad \sum_{i=1}^{p} \nabla h_i(\mathbf{x}) = \mathbf{0}.$$

- Approximate real and virtual diplacements at the nodes:

$$\mathbf{u}^h(\mathbf{x}_j, t) = \sum_{i=1}^{p} h_i(\mathbf{x}_j)\mathbf{q}_i(t) = \mathbf{q}_j(t),$$

$$\boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}_j) = \sum_{i=1}^{p} h_i(\mathbf{x}_j)\boldsymbol{\delta}\mathbf{q}_i = \boldsymbol{\delta}\mathbf{q}_j.$$

Treatment of initial conditions

Since $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$, enforcing $\mathbf{u}^h(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ ensures that the approximate displacement field is consistent with the initial condition. This leads to

$$\mathbf{q}_j(0) = \mathbf{u}^h(\mathbf{x}_j, 0) = \mathbf{u}_0(\mathbf{x}_j).$$

Similarly, imposing the initial velocity condition, $\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$, gives

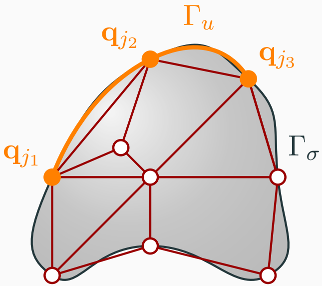
$$\dot{\mathbf{q}}_j(0) = \dot{\mathbf{u}}^h(\mathbf{x}_j, 0) = \mathbf{v}_0(\mathbf{x}_j).$$

✓ These expressions provide the initial values of the unknown vector $\mathbf{q}$ directly in terms of the given initial data $\mathbf{u}_0$ and $\mathbf{v}_0$.

Treatment of boundary conditions

On Γ_u we impose $\mathbf{u}^h = \hat{\mathbf{u}}$ and $\boldsymbol{\delta}\mathbf{u}^h = \mathbf{0}$. Consequently for every node $\mathbf{x}_j \in \Gamma_u$:

$$\mathbf{q}_j(t) = \mathbf{u}^h(\mathbf{x}_j, t) = \hat{\mathbf{u}}(\mathbf{x}_j, t) \quad \text{and} \quad \boldsymbol{\delta}\mathbf{q}_j = \boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}_j, t) = \mathbf{0}.$$



✓ The j-th component of the approximated nodal displacement $\mathbf{q}$ is know for every node $\mathbf{x}_j \in \Gamma_u$.

Semi-discrete weak form of elastodynamics

Given Ω, Γ, $\mathbf{C}$, ρ, $\mathbf{f}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{f}}$, $\mathbf{u}_0$, $\mathbf{v}_0$, and p nodes $\mathbf{x}_1, \dots \mathbf{x}_p$, find the approximated nodal displacements vector $\mathbf{q} \in C^2([0,T], \mathbb{R}^n)$ such that for every $\boldsymbol{\delta}\mathbf{q}$:

$$\boldsymbol{\delta}\mathbf{q}^T \left[\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) - \mathbf{r}(t)\right] = 0$$

coupled with initial conditions

$$\mathbf{q}(0) = \mathbf{u}_0,$$

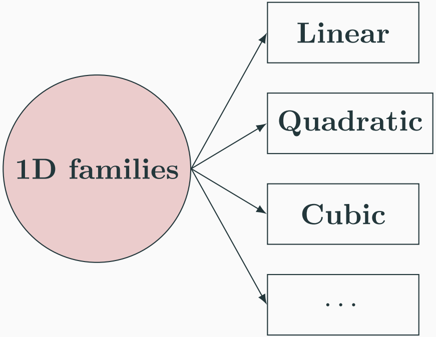
$$\dot{\mathbf{q}}(0) = \mathbf{v}_0.$$

Moreover for every node $\mathbf{x}_j \in \Gamma_u$ we have

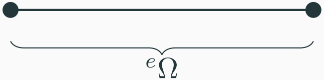
$$\mathbf{q}_j(t) = \hat{\mathbf{u}}(\mathbf{x}_j, t) \quad \text{and} \quad \boldsymbol{\delta}\mathbf{q}_j = \mathbf{0}.$$

1D and 2D elements and shape functions

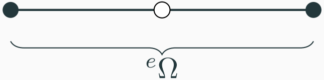
Shape functions for one dimensional structures



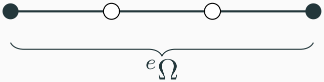
1D finite elements



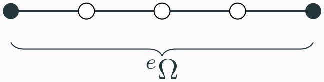
(a) Linear element



(b) Quadratic element

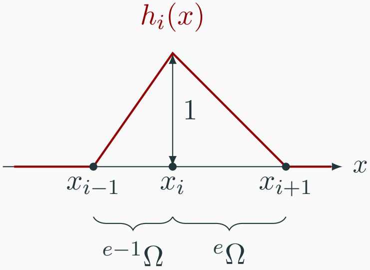


(c) Cubic element



(d) Quartic element

Linear shape functions



$$h_i(x) = \begin{cases} l_2[x_{i-1}, x_i](x) = \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in {}^{e-1}\Omega \\ l_1[x_i, x_{i+1}](x) = \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & x \in {}^{e}\Omega \\ 0 & \text{otherwise} \end{cases}$$

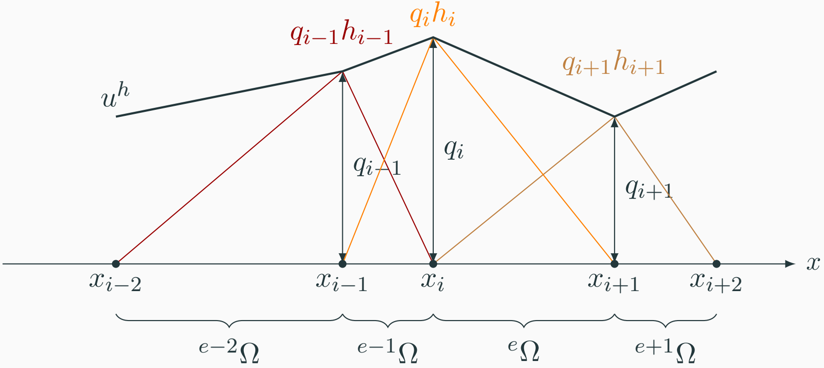
- h_i are piecewise linear functions that, within each finite element, corresponds to a first-degree Lagrange polynomial:

$$l_j[x_1, \ldots, x_n](x) = \prod_{\substack{m=1 \\ m \neq j}}^{n} \frac{x - x_m}{x_j - x_m}$$

- Any linear piecewise function can be expressed as a linear combination of the h_i.

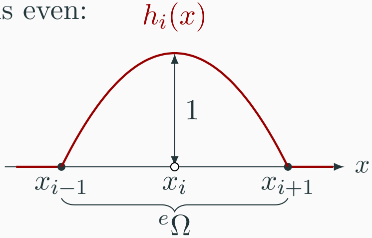
1D displacement approximation via linear shape functions

$$u^h(x,t) = \sum_{i=1}^{p} h_i(x) q_i(t)$$



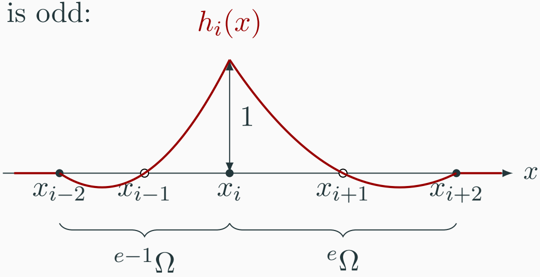
Quadratic shape functions

If i is even:



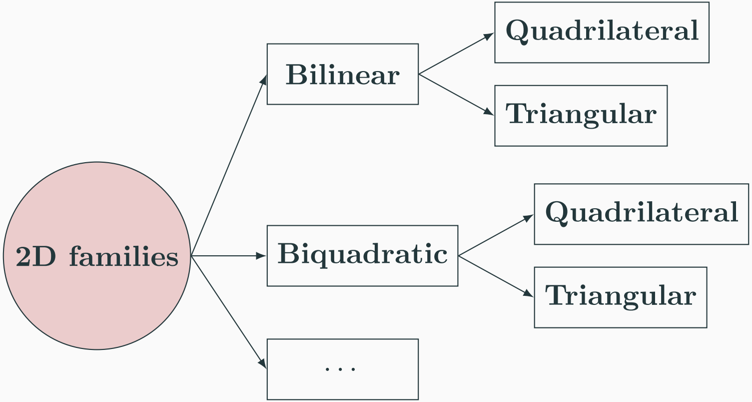
$$h_i(x) = \begin{cases} l_2[x_{i-1}, x_i, x_{i+1}](x) & x \in {}^e\Omega \\ 0 & \text{otherwise} \end{cases}$$

If i is odd:

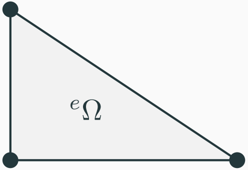


$$h_i(x) = \begin{cases} l_3[x_{i-2}, x_{i-1}, x_i](x) & x \in {}^{e-1}\Omega \\ l_1[x_i, x_{i+1}, x_{i+2}](x) & x \in {}^e\Omega \\ 0 & \text{otherwise} \end{cases}$$

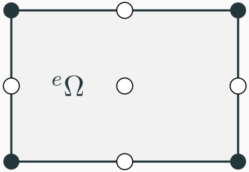
Shape functions for two-dimensional structures



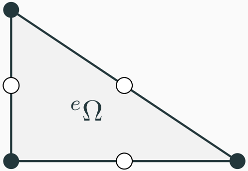
(a) Bilinear Quadrilateral



(b) Bilinear Triangular

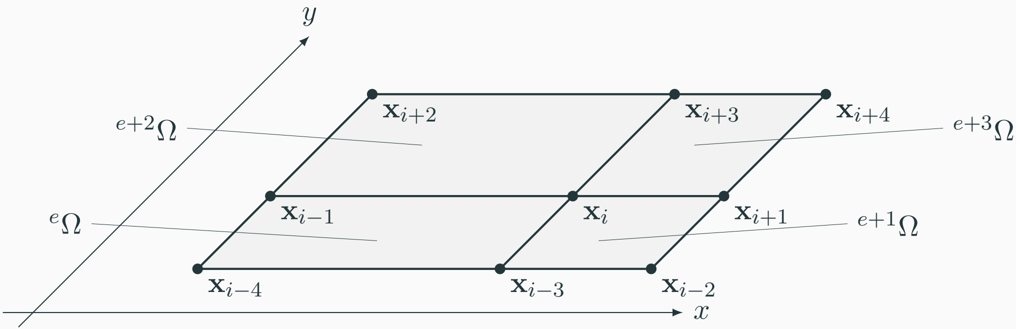


(c) Biquadratic Quadrilateral

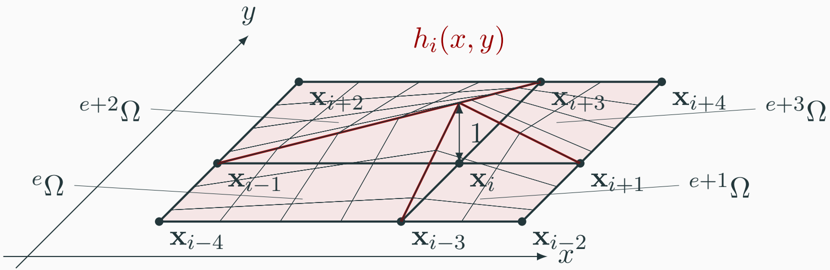


(d) Biquadratic Triangular

Bilinear quadrilateral finite elements mesh

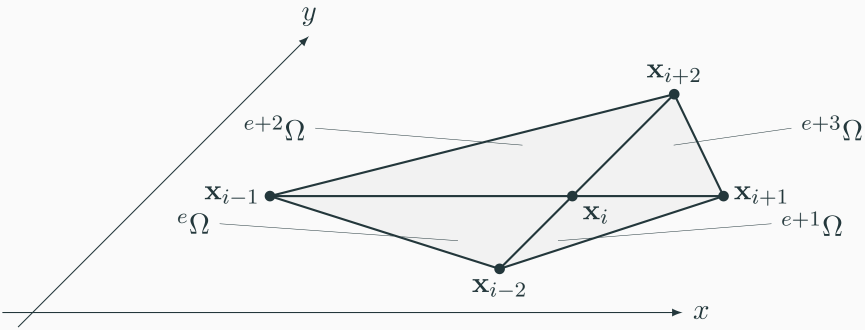


Bilinear quadrilateral shape functions

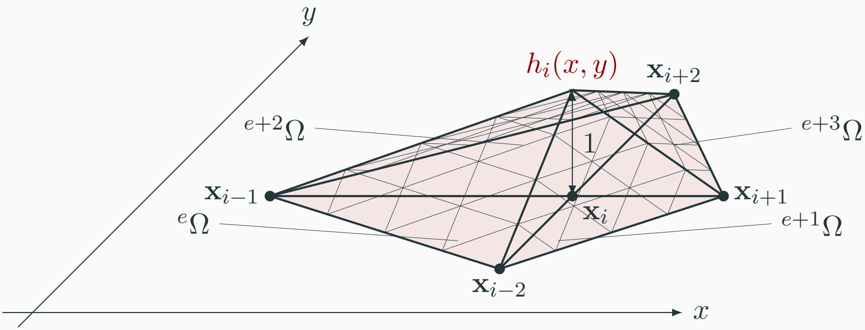


$$h_i(x,y) = \begin{cases} l_2[x_{i-1}, x_i](x) l_2[y_{i-3}, y_i](y) & x \in {}^{e}\Omega \\ l_1[x_i, x_{i+1}](x) l_2[y_{i-3}, y_i](y) & x \in {}^{e+1}\Omega \\ l_2[x_{i-1}, x_i](x) l_1[y_i, y_{i+3}](y) & x \in {}^{e+2}\Omega \\ l_1[x_i, x_{i+1}](x) l_1[y_i, y_{i+3}](y) & x \in {}^{e+3}\Omega \\ 0 & \text{otherwise} \end{cases}$$

Bilinear triangular finite elements mesh



Bilinear triangular shape functions



Advantages and drawbacks of FEM in global coordinate system

Advantages:

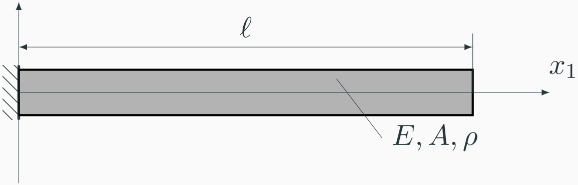
- ✓ The unknowns $\mathbf{q}_i$ have a well-defined physical interpretation, representing the approximate displacement at $\mathbf{x}_i$.
- ✓ Shape functions are systematically defined, ensuring a structured approach to the algorithm.
- ✓ The formulation simplifies the implementation of initial conditions.

Drawbacks:

- ✗ Limited capability in handling complex mesh topologies.
- ✗ Algebraic expressions for shape functions can be computationally cumbersome.
- ✗ The assembly of stiffness and mass matrices is not computationally optimal.

Example: longitudinal vibration of a bar

Example - Finite elements approximation of longitudinal vibrations of a bar



- A (constant) cross-sectional area
- E (constant) Young's modulus (isotropic)
- ρ material density
- ℓ length
- u_1 axial displacement
- x_1 axial coordinate

Objective: determine the first two natural frequencies of the bar using n linear shape functions and compare the results with those obtained from Galerkin's approximation.

Example - Finite elements approximation of longitudinal vibrations of a bar

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