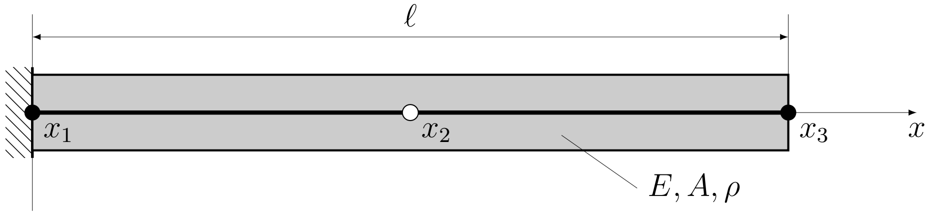
# Problem set 3

## Problem 1

Consider the longitudinal vibrations of a bar of length $l$, with a constant cross-sectional area $A$, constant Young's modulus $E$, and constant material density $\rho$. The bar is fixed at the left end and free at the right end, as shown in the figure below.




The longitudinal vibrations of the bar are governed by the following weak form: find the longitudinal displacement $u_1(\cdot, t) \in \mathcal{U}$, such that the following equation is satisfied for every $\delta u_1 \in \mathcal{V}$

$$\int_0^\ell EA\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial \delta u_1}{\partial x}\right) dx + \int_0^\ell \rho A \ddot{u}_1 \delta u_1 \, dx = 0$$

coupled with the initial conditions expressed in weak or integral form. Moreover,

$$\mathcal{U} = \left\{u_1(\cdot, t) \in H^1(]0, \ell[) \mid u_1(0, t) = 0 \ \forall t \in ]0, T[\right\}$$
$$\mathcal{V} = \left\{\delta u_1 \in H^1(]0, \ell[) \mid \delta u_1(0) = 0\right\}$$

Approximate the first and second natural frequencies using one quadratic finite element and compare the relative error with those obtained by exact analytical solution:

$$\omega_1^{\text{exact}} = \frac{\pi}{2}\sqrt{\frac{E}{\rho l^2}} \qquad \text{and} \qquad \omega_2^{\text{exact}} = \frac{3\pi}{2}\sqrt{\frac{E}{\rho l^2}}.$$

Hint: you can use MATLAB to calculate the coefficients of the stiffness and mass matrices.

## Problem 2

Consider a square rubber membrane, initially at rest at $t = 0$, stretched between fixed supports, and subjected to a distributed transversal load $p$ while experiencing a uniform tension $S$. The weak formulation governing the transverse vibrations of the membrane is: find the transverse displacement $u_3(x, y, t) \in \mathcal{U}$ such that the equation

$$-\int_\Omega S(\nabla \delta u_3)^T \nabla u_3 \, d\Omega + \int_\Omega p \delta u_3 \, d\Omega = \int_\Omega \rho \ddot{u}_3 \delta u_3 \, d\Omega,$$

and the initial conditions

$$\int_\Omega \rho u_3(x,y,0)\delta u_3(x,y)\,dxdy = \int_\Omega \rho \dot{u}_3(x,y,0)\delta u_3(x,y)\,dxdy = 0$$

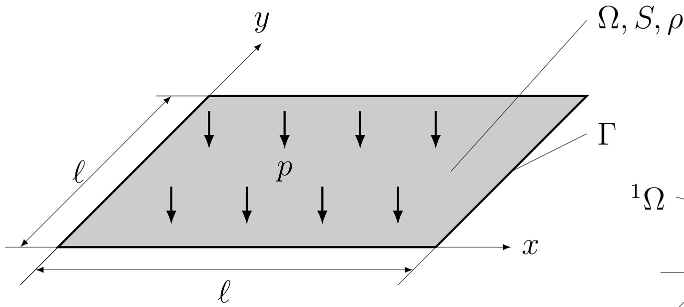
are satisifed for every virtual displacement $\delta u_3 \in \mathcal{V}$. Notice that the intial position and initial velocity of the membrane are both set to zero. Here $\rho$ represents the density, $\Omega = ]0,\ell[\times]0,\ell[$ is the domain (rectangle of sides $a_1$ and $a_2$), and the nabla operator is defined as

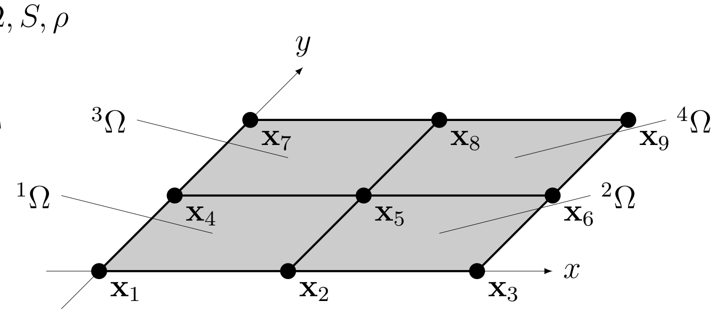
$$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix},$$

The functional spaces are defined as

$$\mathcal{U} = \left\{ u_3(\cdot,t) \in H^1(\Omega); u_3(\cdot,t) = 0 \text{ on } \Gamma \right\},$$
$$\mathcal{V} = \left\{ \delta u_3 \in H^1(\Omega); \delta u_3 = 0 \text{ on } \Gamma \right\},$$

where $\Gamma$ is the boundary of the membrane.




Write a MATLAB script to compute and plot the time response (for a time span of $T = 10$ s) of the geometric center of the membrane, given that the domain is discretized into four bilinear quadrilateral finite elements (with four nodal points per element) and that the distributed load $p$ has a sinusoidal shape with amplitude $A$ and excitation frequency $\bar{\omega}$:

$$p(x,y,t) = A\sin(\bar{\omega}t).$$

Here are the physical parameters of the problem:

- Material: rubber membrane (e.g. silicone rubber),
- Membrane dimensions: $l = 2$ m,
- Membrane tension: $S = 1000$ N/m,
- Mass density: $\rho = 1200$ kg/m$^2$,
- Load amplitude: $A = 100$ N/m$^2$,
- Excitation frequency: $\bar{\omega} = 10\pi$ rad/s.

HINT: use the MATLAB built-in function `dsolve` find the time response at $\mathbf{x}_5$.

Problem 1 is taken from [G] example 3.1.3
Problem 2 is taken from [G] exercise 3.2
[G] Gmür, Dynamique des structures: analyse modale numérique. EPFL Press, 1997