Linear elastodynamics

Galerkin approximation

ME473 Dynamic finite element analysis of structures

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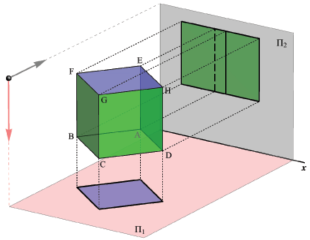
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation

Summary

- Recap week 1
- Further evidence in favor of the weak form
- Discretisation of the weak form of elastodynamics via Galerkin method
- Example: longitudinal vibration of a bar
- Matlab implementation of Galerkin approximation

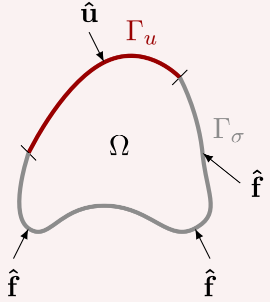
Recommended readings

1. Gmür, Dynamique des structures (§2.3 and §2.4) [GM]
2. Neto et al., Engineering Computation of Structures (§ 2.1 and §2.2) [N]

Recap week 1

Statement of the linear elastodynamics problem

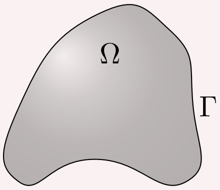
- **Object:**
  A solid $\Omega \subset \mathbb{R}^3$ (beam, shaft, plate etc...) with known material properties: $\mathbf{C}$ and ρ.
- **Main features:**
  - Acting loads on the body: $\mathbf{f}$.
  - Boundary $\Gamma = \Gamma_u \cup \Gamma_\sigma$ (the surface enclosing the solid).
  - Boundary conditions: prescribed displacements $\hat{\mathbf{u}}$ on the boundary Γ_u and/or loads $\hat{\mathbf{f}}$ on the boundary on Γ_σ.
  - Initial displacement $\mathbf{u}_0$ and velocity $\mathbf{v}_0$ at $t = 0$.



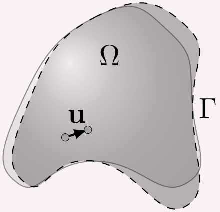
Statement of the linear elastodynamics problem

- **Unknown:** displacement $\mathbf{u} \in C^2(\bar{\Omega} \times [0, T], \mathbb{R}^3)$

Undeformed structure



Deformed structure



- PDE governing the evolution of the displacements **u**:

$$\nabla^T \mathbf{C} \nabla \mathbf{u}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) = \rho \ddot{\mathbf{u}}(\mathbf{x}, t) \qquad \forall (\mathbf{x}, t) \in \Omega \times]0, T[$$

Strong and weak forms of elastodynamics

Strong form

PDE:
$$\nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

BC on Γ_u:
$$\mathbf{u} = \hat{\mathbf{u}}$$

BC on Γ_σ:
$$\mathbf{N}^T \mathbf{C} \nabla \mathbf{u} = \hat{\mathbf{f}}$$

IC at $t = 0$:
$$\mathbf{u} = \mathbf{u}_0, \; \dot{\mathbf{u}} = \mathbf{v}_0$$

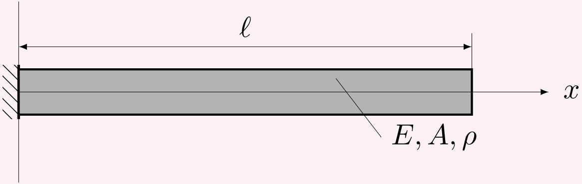
Weak form

Virtual work:
$$\int_\Omega (\nabla \boldsymbol{\delta} \mathbf{u})^T \mathbf{C} \nabla \mathbf{u} \, d\Omega + \int_\Omega \rho \, \boldsymbol{\delta} \mathbf{u}^T \ddot{\mathbf{u}} \, d\Omega = \int_{\Gamma_\sigma} \boldsymbol{\delta} \mathbf{u}^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega \boldsymbol{\delta} \mathbf{u}^T \mathbf{f} \, d\Omega$$

Functional spaces:
$$\mathcal{U} = \{ \mathbf{u}(t, \cdot) \in H^1(\Omega) \mid \mathbf{u} = \hat{\mathbf{u}} \text{ on } \Gamma_u \}$$
$$\mathcal{V} = \{ \boldsymbol{\delta} \mathbf{v} \in H^1(\Omega) \mid \boldsymbol{\delta} \mathbf{v} = \mathbf{0} \text{ on } \Gamma_u \}$$

IC at $t = 0$:
$$\int_\Omega \rho \, \boldsymbol{\delta} \mathbf{u}^T \mathbf{u}|_{t=0} \, d\Omega = \int_\Omega \rho (\boldsymbol{\delta} \mathbf{u})^T \mathbf{u}_0 \, d\Omega,$$
$$\int_\Omega \rho \, \boldsymbol{\delta} \mathbf{u}^T \dot{\mathbf{u}}|_{t=0} \, d\Omega = \int_\Omega \rho (\boldsymbol{\delta} \mathbf{u})^T \mathbf{v}_0 \, d\Omega.$$

Strong form for longitudinal vibrations of a bar



- A cross-sectional area
- E Young's modulus (isotropic bar)
- ρ material density
- ℓ length
- u_1 axial displacement
- x axial coordinate

Find $u_1 \in C^2([0,\ell] \times [0,T])$ such that

$$EA\partial^2_{xx} u_1(x,t) = \rho A \ddot{u}_1(x,t)$$

boundary conditions:

$$u_1(0,t) = 0$$
$$EA\partial_x u_1(\ell,t) = 0$$

initial conditions:

$$u_1(x,0) = u_0(x)$$
$$\dot{u}_1(x,0) = v_0(x)$$

Weak form for longitudinal vibrations of a bar

Find $u_1 \in \mathcal{U}$ such that $\forall \delta u_1 \in \mathcal{V}$ we have

$$\int_0^\ell EA\,\partial_x u_1\,\partial_x(\delta u_1)\;dx + \int_0^\ell \rho A \ddot{u}_1 \delta u_1\,dx = 0,$$

$$\left.\begin{aligned} \int_0^\ell \rho A u(x,0)\delta u_1(x)\,dx &= \int_0^\ell \rho A u_0(x)\delta u_1(x)\,dx, \\ \int_0^\ell \rho A \dot{u}(x,0)\delta u_1(x)\,dx &= \int_0^\ell \rho A v_0(x)\delta u_1(x)\,dx. \end{aligned}\right\}\text{Initial conditions}$$

$$\mathcal{U} = \left\{u_1(\cdot,t) \in H^1(]0,\ell[) \mid u_1(0,t) = 0 \;\forall t \in]0,T[\right\}$$

$$\mathcal{V} = \left\{\delta u_1 \in H^1(]0,\ell[) \mid \delta u_1(0) = 0\right\}$$

Further evidence in favor of the weak form

Weighted residuals

- Let $\mathbf{u}$ be a solution of
$$\nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f} - \rho \ddot{\mathbf{u}} = 0.$$
- Let $\mathbf{u}^h$ an approximate solution:

in general $\mathbf{u}^h$ does not satisfy the differential equation and hence results in an error or a *residual*:
$$\nabla^T \mathbf{C} \nabla \mathbf{u}^h + \mathbf{f} - \rho \ddot{\mathbf{u}}^h = \mathbf{R}^h.$$

- We impose that the residual is zero in a certain Euclidean vector space $\mathcal{V}^h$ of finite dimension. Thus
$$\langle \mathbf{R}^h, \mathbf{v}^h \rangle = 0 \qquad \forall \mathbf{v}^h \in \mathcal{V}^h$$
where $\langle \cdot, \cdot \rangle$ denotes the scalar product.

Weighted residual

- For vector-valued functions, there is a natural definition of scalar product:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_\Omega \mathbf{f}^T \mathbf{g} \, d\Omega.$$

- Hence $\langle \mathbf{R}^h, \mathbf{v}^h \rangle = 0$ for all $\mathbf{v}^h \in \mathcal{V}^h$ implies

$$\int_\Omega (\mathbf{v}^h)^T \left(\nabla^T \mathbf{C} \nabla \mathbf{u}^h + \mathbf{f} - \rho \ddot{\mathbf{u}}^h \right) d\Omega = 0 \qquad \forall \mathbf{v}^h \in \mathcal{V}^h.$$

- Applying the divergence theorem will result into the integral equation of the weak form:

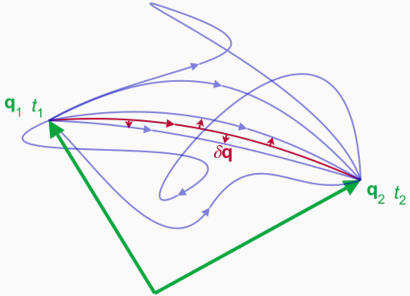
$$\int_\Omega (\nabla \mathbf{v}^h)^T \mathbf{C} \nabla \mathbf{u}^h \, d\Omega + \int_\Omega \rho (\mathbf{v}^h)^T \ddot{\mathbf{u}}^h \, d\Omega = \int_{\Gamma_\sigma} (\mathbf{v}^h)^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega (\mathbf{v}^h)^T \mathbf{f} \, d\Omega \qquad \forall \mathbf{v}^h \in \mathcal{V}^h.$$

Hamilton's principle - principle of stationary action

The path taken by the system is defined by the admissible function $\mathbf{u}$ that makes the functional stationary:

$$J = \int_{t_1}^{t_2} T(\dot{\mathbf{u}}) - U(\mathbf{u}) + W(\mathbf{u})\, dt.$$

- T is the total kinetic energy,
- U represents the potential (elastic) energy of the flexible structure,
- W the work done by external loads that are acting on the body.



Credit: Wikipedia - Hamilton's principle

Hamilton's principle - principle of stationary action.

- The kinetic energy associated with a flexible body that has volume Ω is given as

$$T(\dot{\mathbf{u}}) = \frac{1}{2}\int_{\Omega} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}}\, d\Omega$$

- The total elastic energy of a deformable structure is given as

$$U(\mathbf{u}) = \frac{1}{2}\int_{\Omega} \boldsymbol{\varepsilon}^T \mathbf{C} \boldsymbol{\varepsilon}\, d\Omega$$

  where $\boldsymbol{\varepsilon}$ is the elastic strain and $\mathbf{C}$ is the stiffness material matrix.

- The total work W done by the external mechanical loading is given by

$$W(\mathbf{u}) = \int_{\Omega} \mathbf{u}^T \mathbf{f}\, d\Omega + \int_{\Gamma_\sigma} \mathbf{u}^T \hat{\mathbf{f}}\, d\Gamma$$

Hamilton's principle

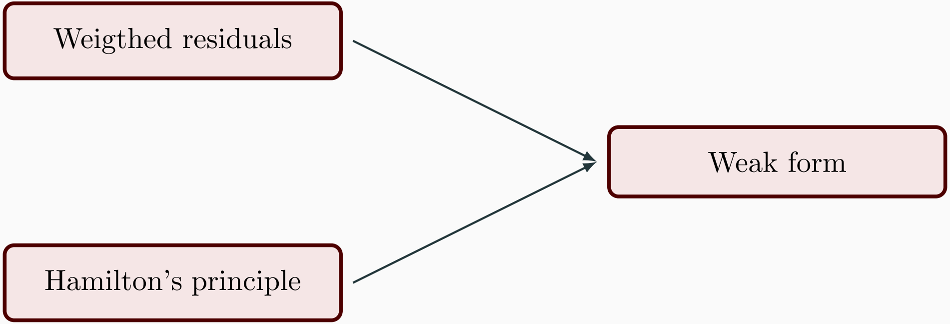
- Computing the first variation δJ, and imposing $\delta J = 0$, yields to:

$$\int_{t_1}^{t_2} \Big[\int_{\Omega} -\rho(\boldsymbol{\delta}\mathbf{u})^T \ddot{\mathbf{u}}\, d\Omega - (\boldsymbol{\delta\varepsilon})^T \mathbf{C}\boldsymbol{\varepsilon} + (\boldsymbol{\delta}\mathbf{u})^T \mathbf{f}\, d\Omega + \int_{\Gamma_\sigma} (\boldsymbol{\delta}\mathbf{u})^T \hat{\mathbf{f}}\, d\Gamma \Big]\, dt = 0$$

for every virtual displacement $\boldsymbol{\delta}\mathbf{u}$ which satisfies $\boldsymbol{\delta}\mathbf{u} = 0$ on Γ_u.
- This implies the weak form of elastodynamics:

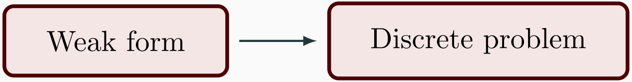
$$\int_{\Omega} (\nabla\boldsymbol{\delta}\mathbf{u})^T \mathbf{C}\nabla\mathbf{u}\, d\Omega + \int_{\Omega} \rho(\boldsymbol{\delta}\mathbf{u})^T \ddot{\mathbf{u}}\, d\Omega = \int_{\Gamma_\sigma} (\boldsymbol{\delta}\mathbf{u})^T \hat{\mathbf{f}}\, d\Gamma + \int_{\Omega} (\boldsymbol{\delta}\mathbf{u})^T \mathbf{f}\, d\Omega \qquad \forall\boldsymbol{\delta}\mathbf{u}.$$

Weak form: cornerstone concept



Galerkin method

General ideas of Galerkin methods



- Galerkin methods are a class of numerical techniques used to transform differential equations in weak formulation, into discrete problems.
- The approximate solution is determined via a finite set of **basis functions**.
- Galerkin approximation compute the best possible approximate solution among a family of potential solutions.

Boris Galerkin 1871 - 1945

Approximate solution of the weak form

Finite-dimensional functional spaces

Choose subspaces $\mathcal{U}^h \subset \mathcal{U}$ and $\mathcal{V}^h \subset \mathcal{V}$ of dimension n and solve the *projected problem* into such subspaces.

Displacement approximation

Instead of searching for $\mathbf{u} \in \mathcal{U}$ such that the weak form is satisfied for any $\boldsymbol{\delta}\mathbf{u} \in \mathcal{V}$, we shall search for $\mathbf{u}^h \in \mathcal{U}^h$ such that the weak form is satisfied for any $\boldsymbol{\delta}\mathbf{u}^h \in \mathcal{V}^h$.

$$\mathbf{u}(\mathbf{x}, t) \approx \mathbf{u}^h(\mathbf{x}, t) \in \mathcal{U}^h \subset \mathcal{U}$$

$$\boldsymbol{\delta}\mathbf{u}(\mathbf{x}) \approx \boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}) \in \mathcal{V}^h \subset \mathcal{V}$$

The key property of the Galerkin approach is that the error is *orthogonal* to the chosen subspaces.

Shape functions

Let $\mathbf{u}^h(\mathbf{x},t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t)$ and $\boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}) = \mathbf{H}(\mathbf{x})\boldsymbol{\delta}\mathbf{q}$ where

- $\mathbf{H}(\mathbf{x})$ is a $3 \times n$ matrix of **shape functions**, defined globally on Ω.
- $\mathbf{q}(t)$ is a $n \times 1$ vector of (*unknown*) time-dependent functions.
- $\boldsymbol{\delta}\mathbf{q}$ is a $n \times 1$ vector of constants.

Shape functions are linearly independent: they form a basis of $\mathcal{U}^h$ and $\mathcal{V}^h$.

$$\begin{pmatrix} u_1^h(\mathbf{x},t) \\ u_2^h(\mathbf{x},t) \\ u_3^h(\mathbf{x},t) \end{pmatrix} = \begin{bmatrix} h_{11}(\mathbf{x}) & h_{12}(\mathbf{x}) & \dots & h_{1n}(\mathbf{x}) \\ h_{21}(\mathbf{x}) & h_{22}(\mathbf{x}) & \dots & h_{2n}(\mathbf{x}) \\ h_{31}(\mathbf{x}) & h_{32}(\mathbf{x}) & \dots & h_{3n}(\mathbf{x}) \end{bmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{pmatrix}$$

$$\begin{pmatrix} \delta u_1^h(\mathbf{x}) \\ \delta u_2^h(\mathbf{x}) \\ \delta u_3^h(\mathbf{x}) \end{pmatrix} = \begin{bmatrix} h_{11}(\mathbf{x}) & h_{12}(\mathbf{x}) & \dots & h_{1n}(\mathbf{x}) \\ h_{21}(\mathbf{x}) & h_{22}(\mathbf{x}) & \dots & h_{2n}(\mathbf{x}) \\ h_{31}(\mathbf{x}) & h_{32}(\mathbf{x}) & \dots & h_{3n}(\mathbf{x}) \end{bmatrix} \begin{pmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_n \end{pmatrix}$$

Approximate solution of the weak form

- Find $\mathbf{u}^h \in \mathcal{U}^h$ such that for all $\boldsymbol{\delta}\mathbf{u}^h \in \mathcal{V}^h$ we have

$$\int_\Omega (\nabla \boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{C} \nabla \mathbf{u}^h \, d\Omega + \int_\Omega \rho (\boldsymbol{\delta}\mathbf{u}^h)^T \ddot{\mathbf{u}}^h \, d\Omega = \int_{\Gamma_\sigma} (\boldsymbol{\delta}\mathbf{u}^h)^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega (\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{f} \, d\Omega$$

- Find $\mathbf{u}^h \in \mathcal{U}^h$ such that for all $\boldsymbol{\delta}\mathbf{q} \in \mathbb{R}^n$ we have

$$\int_\Omega (\nabla \mathbf{H} \boldsymbol{\delta}\mathbf{q})^T \mathbf{C} \nabla \mathbf{H} \mathbf{q}(t) \, d\Omega + \int_\Omega \rho (\mathbf{H} \boldsymbol{\delta}\mathbf{q})^T \mathbf{H} \ddot{\mathbf{q}}(t) \, d\Omega = \int_{\Gamma_\sigma} (\mathbf{H} \boldsymbol{\delta}\mathbf{q})^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega (\mathbf{H} \boldsymbol{\delta}\mathbf{q})^T \mathbf{f} \, d\Omega$$

- Rearranging the terms we obtain the following expression:

$$\boldsymbol{\delta}\mathbf{q}^T \Bigg[\underbrace{\left(\int_\Omega (\nabla \mathbf{H})^T \mathbf{C} \nabla \mathbf{H} \, d\Omega \right)}_{\mathbf{K}} \mathbf{q}(t) + \underbrace{\left(\int_\Omega \rho \, \mathbf{H}^T \mathbf{H} \, d\Omega \right)}_{\mathbf{M}} \ddot{\mathbf{q}}(t) - \underbrace{\left(\int_{\Gamma_\sigma} \mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega \mathbf{H}^T \mathbf{f} \, d\Omega \right)}_{\mathbf{r}(t)} \Bigg] = 0$$

- **Stiffness matrix** $(n \times n)$:

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega$$

where $\mathbf{B}$ is the $(6 \times n)$ deformation matrix defined by $\mathbf{B} = \nabla \mathbf{H}$.

- **Mass matrix** $(n \times n)$:

$$\mathbf{M} = \int_{\Omega} \rho \, \mathbf{H}^T \mathbf{H} \, d\Omega.$$

- **Applied forces vector** $(n \times 1)$:

$$\mathbf{r}(t) = \int_{\Gamma_\sigma} \mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_{\Omega} \mathbf{H}^T \mathbf{f} \, d\Omega.$$

Semi-discrete weak form of elastodynamics

Given Ω, Γ, $\mathbf{C}$, ρ, $\mathbf{f}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{f}}$, $\mathbf{u}_0$, $\mathbf{v}_0$, and a matrix of shape functions $\mathbf{H}$, find the vector $\mathbf{q} \in C^2([0,T], \mathbb{R}^n)$ such that for every vector $\boldsymbol{\delta}\mathbf{q} \in \mathbb{R}^n$ we have

$$\boldsymbol{\delta}\mathbf{q}^T \left[\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) - \mathbf{r}(t)\right] = 0$$

coupled with initial conditions

$$\boldsymbol{\delta}\mathbf{q}^T \left(\mathbf{q}(0) - \mathbf{q_0}\right) = 0,$$

$$\boldsymbol{\delta}\mathbf{q}^T \left(\dot{\mathbf{q}}(0) - \mathbf{p_0}\right) = 0.$$

Treatment of initial conditions (at $t = 0$)

Recall that

$$\int_\Omega \rho(\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{u}\big|_{t=0}\, d\Omega = \int_\Omega \rho(\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{u}_0\, d\Omega$$

$$\int_\Omega \rho(\boldsymbol{\delta}\mathbf{u}^h)^T \dot{\mathbf{u}}\big|_{t=0}\, d\Omega = \int_\Omega \rho(\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{v}_0\, d\Omega$$

Substituting $\mathbf{u}^h(\mathbf{x}, t) = \mathbf{H}(x)\mathbf{q}(t)$ and $\boldsymbol{\delta}\mathbf{u}^h(x) = \mathbf{H}(x)\boldsymbol{\delta}\mathbf{q}$ gives

$$\boldsymbol{\delta}\mathbf{q}^T \Big(\underbrace{\int_\Omega \rho\, \mathbf{H}^T\mathbf{H}\, d\Omega}_{\mathbf{M}}\Big)\mathbf{q}(0) = \boldsymbol{\delta}\mathbf{q}^T \Big(\textstyle\int_\Omega \rho\, \mathbf{H}^T\mathbf{u}_0\, d\Omega\Big)$$

$$\boldsymbol{\delta}\mathbf{q}^T \Big(\underbrace{\int_\Omega \rho\, \mathbf{H}^T\mathbf{H}\, d\Omega}_{\mathbf{M}}\Big)\dot{\mathbf{q}}(0) = \boldsymbol{\delta}\mathbf{q}^T \Big(\textstyle\int_\Omega \rho\, \mathbf{H}^T\mathbf{v}_0\, d\Omega\Big)$$

$$\mathbf{q}(0) = \underbrace{\mathbf{M}^{-1}\Big(\int_\Omega \rho\mathbf{H}^T\mathbf{u}_0\, d\Omega\Big)}_{\mathbf{q_0}}$$

$$\dot{\mathbf{q}}(0) = \underbrace{\mathbf{M}^{-1}\Big(\int_\Omega \rho\mathbf{H}^T\mathbf{v}_0\, d\Omega\Big)}_{\mathbf{p_0}}$$

- **Boundary conditons on** Γ_u:

$$\mathbf{u}^h = \hat{\mathbf{u}} \qquad \text{and} \quad \boldsymbol{\delta}\mathbf{u}^h = \mathbf{0}$$

Consequently, when defining shape functions, it is necessary to impose that

$$\mathbf{H}(\mathbf{x})\mathbf{q}(t) = \hat{\mathbf{u}}(\mathbf{x}, t) \quad \text{and} \qquad \mathbf{H}(\mathbf{x})\boldsymbol{\delta}\mathbf{q} = \mathbf{0} \qquad \forall \mathbf{x} \in \Gamma_u \forall t \in]0, T[.$$

Treatment of boundary conditions

- **Boundary conditons on Γ_u:**

$$\mathbf{u}^h = \hat{\mathbf{u}} \qquad \text{and} \quad \boldsymbol{\delta}\mathbf{u}^h = \mathbf{0}$$

Consequently, when defining shape functions, it is necessary to impose that

$$\mathbf{H}(\mathbf{x})\mathbf{q}(t) = \hat{\mathbf{u}}(\mathbf{x}, t) \quad \text{and} \qquad \mathbf{H}(\mathbf{x})\boldsymbol{\delta}\mathbf{q} = \mathbf{0} \qquad \forall \mathbf{x} \in \Gamma_u \forall t \in]0, T[.$$

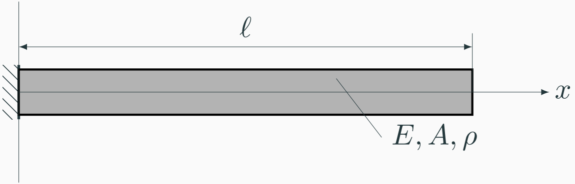
- **Boundary conditon on Γ_σ:**

$$\mathbf{N}^T \mathbf{C} \nabla \mathbf{u}^h = \hat{\mathbf{f}}$$

Does not affect the choice of shape functions since this condition has already been used in the derivation of the weak form.

Example: longitudinal vibration of a bar

Example - Galerkin approximation of longitudinal vibrations of a bar



- A cross-sectional area
- E Young's modulus (isotropic)
- ρ material density
- ℓ length
- u_1 axial displacement
- x axial coordinate

Objective: find a Galerkin approximation of the axial displacement u_1 using appropriate functional spaces.

Galerkin approximation of longitudinal vibrations of a bar

Substituting

$$u_1^h(x,t) = \mathbf{H}(x)\mathbf{q}_1(t) = \sum_{i=1}^{n} h_i(x) q_i(t) \quad \text{and} \quad \delta u_1^h(x) = \mathbf{H}(x)\boldsymbol{\delta}\mathbf{q}_1 = \sum_{i=1}^{n} h_i(x)\delta q_i$$

in the integral equation

$$\int_0^\ell EA\, \partial_x u_1^h\, \partial_x(\delta u_1^h)\; dx + \int_0^\ell \rho A \ddot{u}_1^h \delta u_1^h\, dx = 0,$$

allow us to write

$$\mathbf{K}\mathbf{q}_1(t) + \mathbf{M}\ddot{\mathbf{q}}_1(t) = 0$$

where

$$k_{ij} = \int_0^\ell EA h_i'(x) h_j'(x)\, dx, \qquad m_{ij} = \int_0^\ell \rho A h_i(x) h_j(x)\, dx.$$

Initial and boundary conditions

- **Initial conditions**:

$$u_1^h(x,0) = \mathbf{u}_0(x) \quad \text{and} \quad \dot{u}_1^h(x,0) = \mathbf{v}_0(x)$$

Thus $\mathbf{q}_1(0) = \mathbf{q}_0 = \mathbf{M}^{-1} \int_\Omega \rho \mathbf{H}^T \mathbf{u}_0 d\Omega$ and $\dot{\mathbf{q}}_1(0) = \mathbf{p}_0 = \mathbf{M}^{-1} \int_\Omega \rho \mathbf{H}^T \mathbf{v}_0 d\Omega$.

Initial and boundary conditions

- **Initial conditions**:

$$u_1^h(x,0) = \mathbf{u}_0(x) \quad \text{and} \quad \dot{u}_1^h(x,0) = \mathbf{v}_0(x)$$

Thus $\mathbf{q}_1(0) = \mathbf{q}_0 = \mathbf{M}^{-1} \int_\Omega \rho \mathbf{H}^T \mathbf{u}_0 d\Omega$ and $\dot{\mathbf{q}}_1(0) = \mathbf{p}_0 = \mathbf{M}^{-1} \int_\Omega \rho \mathbf{H}^T \mathbf{v}_0 d\Omega$.

- **Boundary conditions**:

$$u_1^h(0,t) = 0 \quad \text{and} \quad \delta u_1^h(0,t) = 0$$

Therefore $\mathbf{H}(0) = \mathbf{0}$.

One-term Galerkin approximation

- Let $n = 1$ then

$$u_1^h(x,t) = \mathbf{H}(x)\mathbf{q}_1(t) = h_1(x)q_1(t)$$
$$\delta u_1^h(x) = \mathbf{H}(x)\boldsymbol{\delta}\mathbf{q}_1 = h_1(x)\delta q_1$$

where we choose

$$h_1(x) = \frac{x}{\ell}.$$

Notice that $h_1 \in H^1(]0,\ell[)$ and $h_1(0) = 0$.

- The semi-discrete weak form is: find the function $q_1(t)$ such that:

$$k_{11}q_1(t) + m_{11}\ddot{q}_1(t) = 0$$

$$k_{11} = \int_0^\ell EA\left(\frac{1}{\ell}\right)^2 dx = \frac{EA}{\ell}, \qquad m_{11} = \int_0^\ell \rho A\left(\frac{x}{\ell}\right)^2 dx = \frac{\rho A\ell}{3}$$

Second order differential equation

The linear homogeneous second order ODE with constant coefficients governing the free vibration of single degree of freedom conservative system:

$$\begin{cases} k_{11} q_1(t) + m_{11} \ddot{q}_1(t) = 0 \\ q_1(0) = q_0 \\ \dot{q}_1(0) = p_0 \end{cases}$$

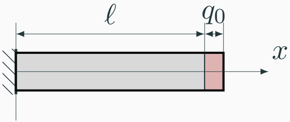
admits an unique solution given by

$$q_1(t) = p \cos(\omega_1 t - \varphi)$$

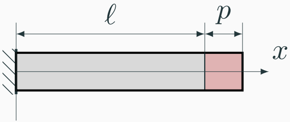
where $\omega_1 = \sqrt{k_{11}/m_{11}}$, $p = \sqrt{q_0^2 + (p_0/\omega_1)^2}$ and $\tan(\varphi) = p_0/q_0\omega_1$.

Displacement approximation

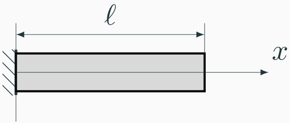
$$u_1(x,t) = h_1(x)q_1(t) = \frac{px}{\ell}\cos(\omega_1 t - \varphi)$$



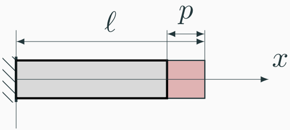
(a) $t = 0$



(b) $t = (\varphi + 2k\pi)/\omega_1$



(c) $t = (\varphi + \pi/2 + k\pi)/\omega_1$



(d) $t = (\varphi + (2k+1)\pi)/\omega_1$

Fundamental frequency comparison

- Approximated fundamental frequency: (obtained via one-term Galerkin)

$$\omega_1 = \sqrt{\frac{k_{11}}{m_{11}}} = \sqrt{3}\sqrt{\frac{E}{\rho l^2}}$$

- Exact fundamental frequency:

$$\omega_1^e = \frac{\pi}{2}\sqrt{\frac{E}{\rho \ell^2}}$$

- Relative error:

$$\text{relative error} = \frac{|\omega_1 - \omega_1^e|}{\omega_1^e} = 10.3\%$$

One-term quadratic Galerkin approximation

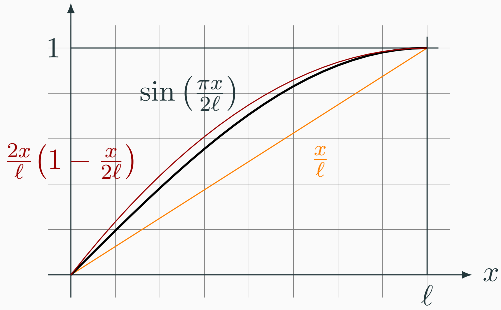
Let $n = 1$ and

$$u_1^h(x,t) = \mathbf{H}(x)\mathbf{q}_1(t) = h_1(x)q_1(t)$$
$$\delta u_1^h(x) = \mathbf{H}(x)\boldsymbol{\delta}\mathbf{q}_1 = h_1(x)\delta q_1$$

where the shape function is choosen as

$$h_1(x) = \frac{2x}{\ell}\left(1 - \frac{x}{2\ell}\right).$$

Notice $h_1 \in H^1(]0,\ell[)$ and $h_1(0) = 0$.



One-term quadratic Galerkin approximation

- The stiffness and mass '*matrices*' are

$$k_{11} = \int_0^{\ell} \frac{4EA}{\ell^2}\left(1 - \frac{x}{\ell}\right)^2 dx = \frac{4EA}{3\ell}$$

$$m_{11} = \int_0^{\ell} \frac{4\rho A x^2}{\ell^2}\left(1 - \frac{x}{2\ell}\right)^2 dx = \frac{8\rho A\ell}{15}$$

- Approximated fundamental frequency: (obtained via one-term quadratic Galerkin)

$$\omega_1 = \sqrt{\frac{k_{11}}{m_{11}}} = \sqrt{\frac{5}{2}}\sqrt{\frac{E}{\rho\ell^2}}$$

- Relative error:

$$\text{relative error} = \frac{|\omega_1 - \omega_1^e|}{\omega_1^e} = 0.7\%$$

n-terms Galerkin approximation

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Advantages and drawbacks of Galerkin method

In a nutshell: Galerkin method transforms the partial differential equations (PDE), expressed in their weak formulation, into a system of ordinary differential equations (ODE).

Advantages:

- ✓ Converges quickly with appropriate shape functions.
- ✓ Provides a systematic and structured approach for approximating solutions.
- ✓ The same set of functions is used to express real and virtual variables.

Drawbacks:

- ✗ Accuracy heavily dependent on choice of basis functions.
- ✗ No physical interpretation of the unknown variable $\mathbf{q}(t)$.
- ✗ The formulation of initial and boundary conditions in the discretized form is cumbersome.