

# Eigenvalue problems of vibrations and stability

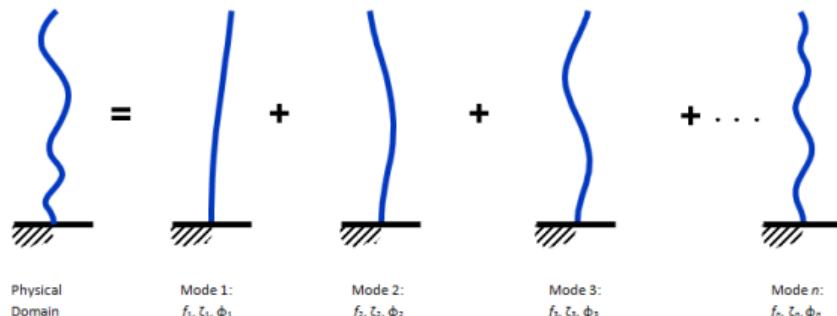
Analysis of free and forced vibrations

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ME473 Dynamic finite element analysis of structures

Stefano Burzio

2025



# Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation
3		FEM global	Project 1 statement
4		FEM local	
5		FEM local	Project 1 submission
6	Classical structural elements	Bars and trusses	Project 2 statement
7		Beams	
8		Frames and grids	
9		Kirchhoff-Love plates	Project 2 submission
10		Kirchhoff-Love plates	Project 3 statement
11		Reissner-Mindlin plates	
12	Free and forced vibrations	Analysis of free vibrations	

## Summary

- Recap week 11
- Modal properties of conservative systems
- Example: Rayleigh's quotient for axial vibrations of a free-free bar
- Numerical modal extraction algorithms for conservative systems

## Recommended readings

(N) Neto et al., Engineering Computation of Structures (chap. 2.5)

(P) Petyt, Introduction to finite element vibration analysis (chap. 11)

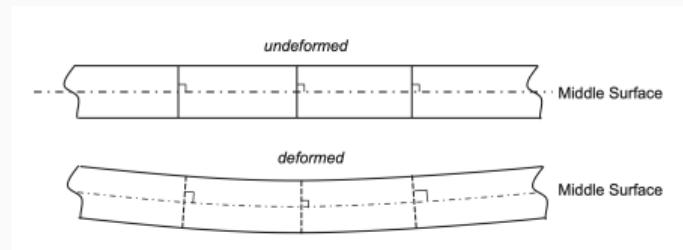
(G) Gmür, Dynamique des structures (§4.1 and §4.2)

## Recap week 11

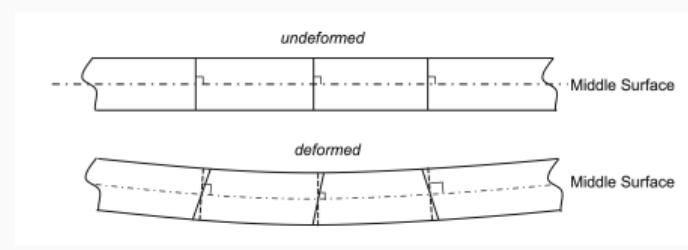
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# Plates theories

## Thin plates



## Thick plates



- Planarity and orthogonality of the cross-sectional planes.
- Displacement:

$$\mathbf{u} = [-z\partial_x u_3 \quad -z\partial_y u_3 \quad u_3]^T$$

- Planarity of the cross-sectional planes.
- Displacement:

$$\mathbf{u} = [z\phi_2 \quad -z\phi_1 \quad u_3]^T$$

## Strong form for Reissner-Mindlin plate bending

Let  $\Omega = [-a, a] \times [-b, b]$  be a rectangular plate. Find the transverse displacement  $u_3 \in C^2(\Omega \times [0, T])$  and the rotations  $\phi_1, \phi_2 \in C^2(\Omega \times [0, T])$  such that

$$\nabla_m^T \bar{C} \nabla_r \mathbf{u} + \mathbf{f} = \mathbf{I} \ddot{\mathbf{u}} \quad \text{on } \Omega \times ]0, T[$$

- boundary conditions (simply supported):

$$u_3 = 0 \quad \text{in } \partial\Omega \times ]0, T[$$

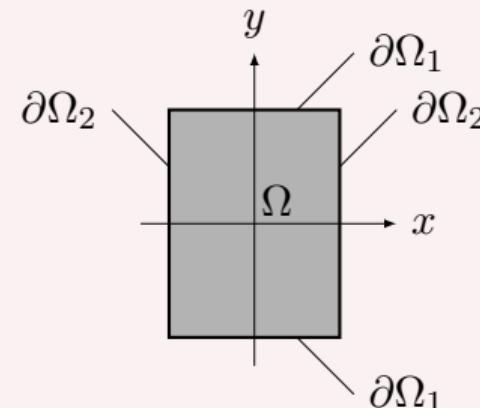
$$\phi_2 = 0 \quad \text{in } \partial\Omega_1 \times ]0, T[$$

$$\phi_1 = 0 \quad \text{in } \partial\Omega_2 \times ]0, T[$$

- initial conditions:

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in } \Omega$$

$$\dot{\mathbf{u}}(\cdot, 0) = \mathbf{v}_0 \quad \text{in } \Omega$$



## Definitions

- Reissner-Mindlin differential operators:

$$\nabla_r = \begin{bmatrix} \nabla_b \\ \nabla_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & -\partial_y & 0 \\ 0 & -\partial_x & \partial_y \\ \partial_x & 0 & 1 \\ \partial_y & -1 & 0 \end{bmatrix}, \quad \text{and} \quad \nabla_m = \begin{bmatrix} \nabla_b \\ \nabla_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & -\partial_y & 0 \\ 0 & -\partial_x & \partial_y \\ \partial_x & 0 & -1 \\ \partial_y & 1 & 0 \end{bmatrix}.$$

- Constitutive matrix:  $\bar{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{C}}_b & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{C}}_s \end{bmatrix}$

$$\bar{\mathbf{C}}_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{C}}_s = \frac{Ekh}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Mass moment of inertia matrix:  $\mathbf{I} = \begin{bmatrix} \rho h & 0 & 0 \\ 0 & \rho h^3/12 & 0 \\ 0 & 0 & \rho h^3/12 \end{bmatrix}$

## Generalized displacements approximation

Since the rotations  $\phi_1$  and  $\phi_2$  are defined independently of the transversal displacement  $u_3$ , the discretization procedure uses 2D bilinear finite elements.

$${}^e\mathbf{u}^h(\mathbf{x}, t) = {}^e\mathbf{H}(\mathbf{x}) {}^e\mathbf{q}(t) = \sum_{i=1}^4 {}^e h_i(\mathbf{x}) {}^e \mathbf{q}^i(t)$$

- ${}^e\mathbf{H}(\mathbf{x})$  is a  $3 \times 12$  matrix of **shape functions** ( $\mathbf{I}$  is the  $3 \times 3$  identity matrix.):

$${}^e\mathbf{H} = \left[ \begin{array}{|c|c|c|c|} \hline h_1\mathbf{I} & h_2\mathbf{I} & h_3\mathbf{I} & h_4\mathbf{I} \\ \hline \end{array} \right] = \begin{bmatrix} {}^e h_1 & 0 & 0 & {}^e h_4 & 0 & 0 \\ 0 & {}^e h_1 & 0 & \dots & 0 & {}^e h_4 \\ 0 & 0 & {}^e h_1 & & 0 & 0 \\ & & & & 0 & {}^e h_4 \end{bmatrix}$$

- ${}^e \mathbf{q}^i(t) = \begin{bmatrix} {}^e d^i(t) \\ {}^e \theta_1^i(t) \\ {}^e \theta_2^i(t) \end{bmatrix}$  and  ${}^e \mathbf{u}^h = \begin{bmatrix} {}^e u_3 \\ {}^e \phi_1 \\ {}^e \phi_2 \end{bmatrix}$  are the generalized displacements of node  $i$  and inside the element  $e$  respectively.

## Elementary matrices and loads vector

- Elementary stiffness matrix ( $12 \times 12$ ):

$${}^e\mathbf{K} = \underbrace{\int_{{}^e\Omega} {}^e\mathbf{B}_b^T \overline{\mathbf{C}}_b {}^e\mathbf{B}_b d\Omega}_{{}^e\mathbf{K}_b} + \underbrace{\int_{{}^e\Omega} {}^e\mathbf{B}_s^T \overline{\mathbf{C}}_s {}^e\mathbf{B}_s d\Omega}_{{}^e\mathbf{K}_s}$$

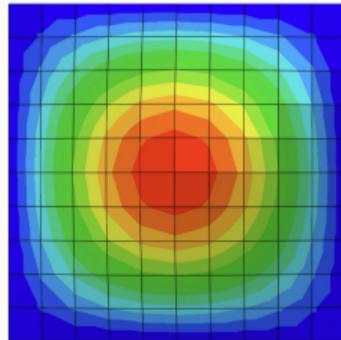
- Elementary mass matrix ( $12 \times 12$ ):

$${}^e\mathbf{M} = \int_{{}^e\Omega} {}^e\mathbf{H}^T \mathbf{I} {}^e\mathbf{H} d\Omega.$$

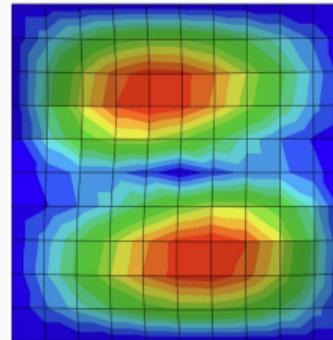
- Elementary applied forces vector ( $12 \times 1$ ):

$${}^e\mathbf{r}(t) = \int_{{}^e\Omega} {}^e\mathbf{H}^T \mathbf{f} d\Omega.$$

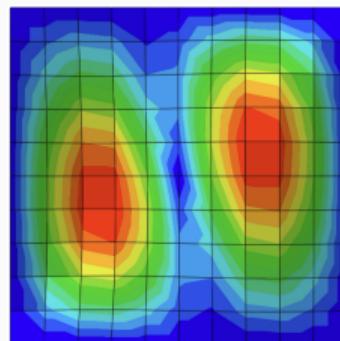
## Addendum: modal shapes for SSSS thick plates (span/thickness = 0.1)



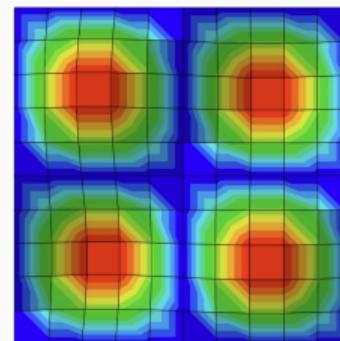
(a) Mode shape 1:  $\omega_{11}^h = 9.672$  Hz



(b) Mode shape 2:  $\omega_{12}^h = 23.643$  Hz

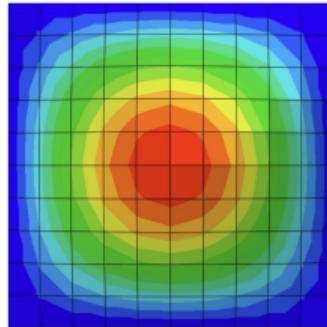


(c) Mode shape 3:  $\omega_{21}^h = 23.643$  Hz

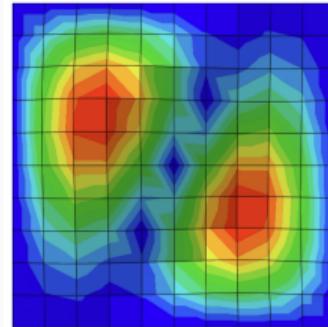


(d) Mode shape 4:  $\omega_{22}^h = 36.114$  Hz

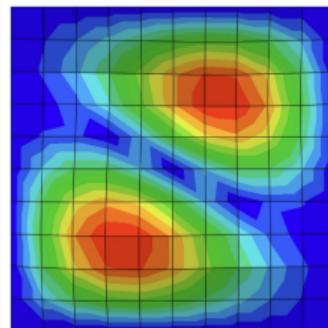
## Addendum: modal shapes for SSSS thin plates (span/thickness = 0.01)



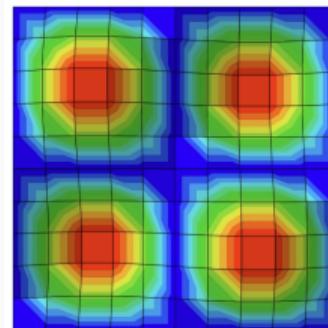
(a) Mode shape 1:  $\omega_{11}^h = 1.001$  Hz



(b) Mode shape 2:  $\omega_{12}^h = 2.573$  Hz



(c) Mode shape 3:  $\omega_{21}^h = 2.573$  Hz



(d) Mode shape 4:  $\omega_{22}^h = 4.103$  Hz

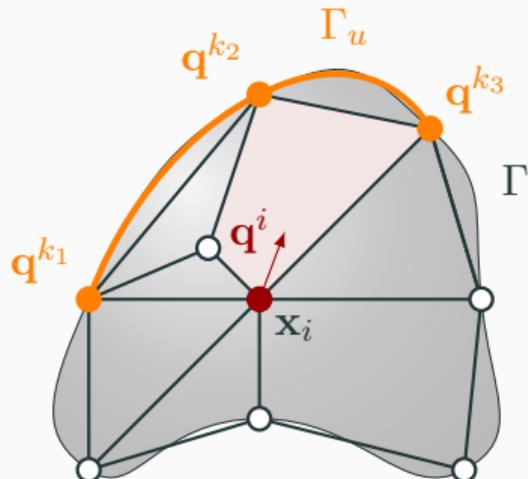
## Analysis of free vibrations

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# Free vibrations of non-rotating conservative systems

The discretization of linear three-dimensional elastodynamics, as well as the analysis of vibrations in beams and plates via FEM, all lead to a system of ODE:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t),$$



**Free vibration:** no external forcing is applied, i.e.  $\mathbf{r}(t) = \mathbf{0}$ .

- Generalized nodal displacements:  
$$\mathbf{q}(t) = [\mathbf{q}^1(t), \dots, \mathbf{q}^n(t)]^T.$$
- **Boundary conditions:**  $\mathbf{q}^k = \hat{\mathbf{q}}^k$  for all  $k$  such that  $\mathbf{x}_k \in \Gamma_u$ .
- **Initial conditions:**  $\mathbf{q}(0) = \mathbf{u}_0$  and  $\dot{\mathbf{q}}(0) = \mathbf{v}_0$

## Harmonic response

The solution for a free undamped discrete vibration problem

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}$$

is sinusoidal, described via an harmonic function:

$$\mathbf{q}(t) = \alpha \mathbf{p} \cos(\omega t + \varphi)$$

- $\mathbf{p}$ : mode shape (modal vector)
- $\omega$ : natural frequency
- $\varphi$ : phase
- $\alpha$ : scaling factor

- **Solutions are defined up to a scalar factor:  $\alpha$ .**
- Three scalars unknowns  $\alpha, \omega, \varphi$  and one vector unknown  $\mathbf{p}$ .

## Generalized eigenvalue problem

Substituting the proposed ansatz into the equation of motion yields a generalized eigenvalue problem:

$$\begin{aligned}\alpha(\mathbf{K} - \omega^2\mathbf{M})\mathbf{p} \cos(\omega t - \varphi) &= 0 \\ (\mathbf{K} - \omega^2\mathbf{M})\mathbf{p} &= 0\end{aligned}$$

### Solving the eigenvalue problem:

- Eigenvalue:  $\lambda_j = \omega_j^2$  are the roots of the characteristic polynomial:

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0.$$

- Eigenvector:  $\mathbf{p}_j$  are the solution of the equation

$$(\mathbf{K} - \lambda_j \mathbf{M})\mathbf{p}_j = \mathbf{0}.$$

## General solution via modal superposition

- Once the set of eigenvalue-eigenvector pairs  $(\omega_j^2, \mathbf{p}_j)$  have been determined, the linearity of the system allows the general solution to be expressed as a superposition of modal contributions:

$$\mathbf{q}(t) = \sum_{j=1}^n \alpha_j \mathbf{p}_j \cos(\omega_j t + \varphi_j)$$

- The constants  $\alpha_j$  and  $\varphi_j$  are determined from the initial conditions:  $\mathbf{q}(0) = \mathbf{u}_0$  and  $\dot{\mathbf{q}}(0) = \mathbf{v}_0$  leading to the system:

$$\sum_{j=1}^n \alpha_j \mathbf{p}_j \cos(\varphi_j) = \mathbf{u}_0$$

$$\sum_{j=1}^n \alpha_j \omega_j \mathbf{p}_j \sin(\varphi_j) = -\mathbf{v}_0$$

## Fundamental properties of generalized eigenvalue problem

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## Spectral property

- If both  $\mathbf{K}$  and  $\mathbf{M}$  are symmetric and strictly positive definite, then all eigenvalues  $\lambda_j$  of the generalized eigenvalue problem

$$\mathbf{K}\mathbf{p}_j = \lambda_j \mathbf{M}\mathbf{p}_j$$

are real and positive.

- Let  $\omega_j = \sqrt{\lambda_j}$ , then

$$0 < \omega_1 \leq \omega_2 \leq \omega_3 \leq \cdots \leq \omega_n.$$

## Rigid body modes

In the semi-discrete weak form obtained via finite element discretization:

- The **mass matrix  $M$**  is symmetric and strictly positive definite.
- The **stiffness matrix  $K$**  is symmetric and positive semi-definite:

$$Kp = 0 \quad \text{for certain nonzero vectors } p.$$

Consequently, the eigenvalues  $\omega_j^2$  of the generalized eigenvalue problem are all real and non-negative:

$$0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_n.$$

**Rigid body modes:** zero eigenvalues (i.e.,  $\omega_j = 0$ ) correspond to *rigid body motions*, where the system undergoes displacement without internal deformation.

## Spectral offset technique

To ensure that the stiffness matrix is *strictly* positive definite, a **spectral shift** strategy is employed. Given  $\sigma > 0$ , define the modified stiffness matrix as:

$$\mathbf{K}_\sigma = \mathbf{K} + \sigma \mathbf{M}$$

- Consider the offset eigenvalue problem:  $(\mathbf{K}_\sigma - \lambda_\sigma \mathbf{M})\mathbf{p} = \mathbf{0}$
- Substituting the definition of  $\mathbf{K}_\sigma$ :

$$(\mathbf{K} + \sigma \mathbf{M} - \lambda_\sigma \mathbf{M})\mathbf{p} = (\mathbf{K} + (\sigma - \lambda_\sigma) \mathbf{M})\mathbf{p} = \mathbf{0}$$

- Comparing with  $(\mathbf{K} - \lambda \mathbf{M})\mathbf{p} = \mathbf{0}$ , we identify:  $\lambda_\sigma = \lambda + \sigma > 0$ .

**Eigenvalues are shifted by  $\sigma$ , but the eigenvectors remain the same.**

**Empirical rule:** 
$$\sigma \approx \frac{1}{100} \frac{\text{tr}(\mathbf{K})}{\text{tr}(\mathbf{M})}$$

## Orthogonality of mode shapes

Let  $\mathbf{p}_i$  and  $\mathbf{p}_j$  two eigenvectors corresponding to two *distinct* eigenvalues  $\lambda_i$  and  $\lambda_j$ , then

$$\mathbf{p}_i^T \mathbf{M} \mathbf{p}_j = 0 \quad \text{and} \quad \mathbf{p}_i^T \mathbf{K} \mathbf{p}_j = 0 \quad (i \neq j).$$

**Consequence:** two different harmonic responses:

$$\mathbf{q}_i(t) = \alpha_i \mathbf{p}_i \cos(\omega_i t + \varphi_i) \quad \text{and} \quad \mathbf{q}_j(t) = \alpha_j \mathbf{p}_j \cos(\omega_j t + \varphi_j)$$

are  $\mathbf{M}$ - and  $\mathbf{K}$ -orthogonal:

$$\mathbf{q}_i^T \mathbf{M} \mathbf{q}_j = 0 \quad \text{and} \quad \mathbf{q}_i^T \mathbf{K} \mathbf{q}_j = 0.$$

The virtual work of inertial and elastic forces of a given mode, along the displacement given by a different mode, is zero.

## Normalization of mode shapes

Let  $\mathbf{p}_i$  and  $\mathbf{p}_j$  two eigenvectors corresponding to the eigenvalues  $\lambda_i$  and  $\lambda_j$ , then

$$\mathbf{p}_i^T \mathbf{M} \mathbf{p}_j = \mu_i \delta_{ij} \quad \text{and} \quad \mathbf{p}_i^T \mathbf{K} \mathbf{p}_j = \kappa_i \delta_{ij}$$

where

- $\mu_i$  = modal mass of mode  $i$ ,
- $\kappa_i$  = modal stiffness of mode  $i$ ,
- $\delta_{ij}$  = Kronecker delta.

**Usual normalization: (orthonormalization):**

$$\mu_i = 1 \quad \Rightarrow \quad \lambda_i = \frac{\kappa_i}{\mu_i} = \omega_i^2.$$

## Orthonormalization of mode shapes

Let  $\mathbf{p}_i$  and  $\mathbf{p}_j$  two eigenvectors corresponding to the eigenvalues  $\lambda_i$  and  $\lambda_j$ , then

$$\mathbf{p}_i^T \mathbf{M} \mathbf{p}_j = \delta_{ij} \quad \text{and} \quad \mathbf{p}_i^T \mathbf{K} \mathbf{p}_j = \omega_i^2 \delta_{ij}$$

where  $\delta_{ij}$  represent Kronecker symbol.

**Consequences:** if we organize the modal vectors  $\mathbf{p}_i$  in a so-called modal matrix  $\mathbf{P}$ :

$$\mathbf{P} = [ \mathbf{p}_1 \mid \mathbf{p}_2 \mid \dots \mid \mathbf{p}_n ]$$

then

$$\mathbf{P}^T \mathbf{M} \mathbf{P} = \mathbf{I} \quad \text{and} \quad \mathbf{P}^T \mathbf{K} \mathbf{P} = \mathbf{\Lambda}$$

where  $\mathbf{I}$  is the identity matrix of order  $n$  and  $\mathbf{\Lambda}$  the spectral matrix:

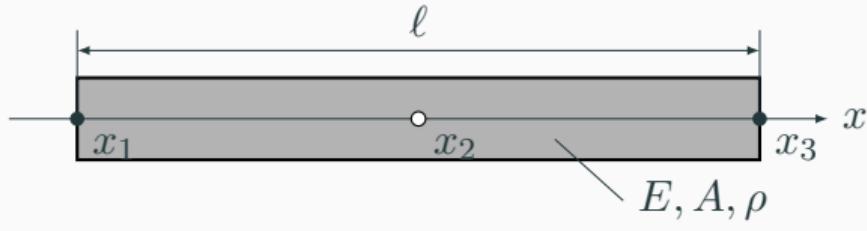
$$\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n) = \text{diag}(\omega_1^2, \dots, \omega_n^2).$$

Example: longitudinal vibrations of  
free-free bar

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# Longitudinal vibrations of free-free bar

Consider a free-free bar discretized by one bilinear element.



- $\ell$  length
- $A$  cross-sectional area
- $E$  Young's modulus
- $\rho$  material density

## Objectives:

- ① compute the approximate natural frequencies and corresponding mode shapes,
- ② verify the presence of a rigid body mode,
- ③ check the orthonormality of the mode shapes.

# MATLAB example - longitudinal vibrations of free-free bar

▶ Go to Matlab Drive

## Eigenproblem solution methods

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# Transformation to standard form by inversion

Generalized  
eigenvalue problem

$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{p} = 0$$

$$\begin{matrix} \text{detM} \neq 0 \\ \text{detK} \neq 0 \end{matrix}$$

Standard eigenvalue problem

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{p} = 0$$

$\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$  dynamic stiffness matrix

Standard eigenvalue problem

$$(\mathbf{A} - \theta \mathbf{I})\mathbf{p} = 0$$

$\mathbf{A} = \mathbf{K}^{-1}\mathbf{M}$  dynamic flexibility matrix  
 $\theta = 1/\lambda$

The matrices  $\mathbf{M}^{-1}\mathbf{K}$  and  $\mathbf{K}^{-1}\mathbf{M}$  are not symmetric.

## Transformation to standard form by decomposition

To reduce to a standard eigenvalue problem:

- Let  $\mathbf{L}$  be the *Cholesky factor* of  $\mathbf{M}$ :  $\mathbf{M} = \mathbf{LL}^T$ .

Since  $\mathbf{M}$  is symmetric positive-definite,  $\mathbf{L}$  is a real lower triangular matrix with positive diagonal entries.

- Define the change of variables  $\mathbf{v} = \mathbf{L}^T \mathbf{p}$ . Substituting into the original problem and multiplying by  $\mathbf{L}^{-1}$  yields:

$$(\mathbf{K} - \omega^2 \mathbf{LL}^T) \mathbf{p} = 0 \quad \Rightarrow \quad (\mathbf{L}^{-1} \mathbf{KL}^{-T} - \omega^2 \mathbf{I}) \mathbf{v} = 0.$$

- Result:** standard eigenvalue problem for symmetric and positive definite matrix  $\mathbf{A} = \mathbf{L}^{-1} \mathbf{KL}^{-T}$ :

$$(\mathbf{A} - \lambda \mathbf{I}) \mathbf{v} = \mathbf{0}.$$

## Transformation to standard form by decomposition

- Cholesky decomposition can be applied to  $\mathbf{K}$  as well, since  $\mathbf{K}$  is symmetric positive-definite:

$$\mathbf{K} = \mathbf{L}\mathbf{L}^T.$$

- **Result:** standard eigenvalue problem for symmetric and positive definite matrix  $\mathbf{A} = \mathbf{L}^{-1}\mathbf{M}\mathbf{L}^{-T}$ , eigenvalues  $\theta = 1/\lambda$  and eigenvectors  $\mathbf{v} = \mathbf{L}^T\mathbf{p}$ :

$$(\mathbf{A} - \theta\mathbf{I})\mathbf{v} = \mathbf{0}.$$

# Small and large scale problems

## Small-scale problems

- system matrices are of modest size  $n \leq 250$  or  $250 \leq n \leq 2500$  and small bandwidth matrices.
- An explicit reduction to standard eigenvalue form is typically employed.

## Large-scale problems

- $n \geq 2500$ .
- The most effective algorithms are subspace iteration, simulation iteration, Lanczos' method, Arnoldi method and Davidson method.

## Numerical modal extraction algorithms

### 1. Rayleigh's quotient

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# Definition and first properties

Let  $\mathbf{w}$  be a vector

$$\mathcal{R}(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{K} \mathbf{w}}{\mathbf{w}^T \mathbf{M} \mathbf{w}}$$

## Properties

① Homogeneity. Let  $\alpha$  a non-zero constant, then

$$\mathcal{R}(\alpha \mathbf{w}) = \frac{(\alpha \mathbf{w})^T \mathbf{K} (\alpha \mathbf{w})}{(\alpha \mathbf{w})^T \mathbf{M} (\alpha \mathbf{w})} = \frac{\alpha^2 \mathbf{w}^T \mathbf{K} \mathbf{w}}{\alpha^2 \mathbf{w}^T \mathbf{M} \mathbf{w}} = \mathcal{R}(\mathbf{w})$$

② Rayleigh quotient of an eigenvector:

$$\mathcal{R}(\mathbf{w} = \mathbf{p}_i) = \frac{\mathbf{p}_i^T \mathbf{K} \mathbf{p}_i}{\mathbf{p}_i^T \mathbf{M} \mathbf{p}_i} = \omega_i^2$$

## Stationarity of Rayleigh's quotient

First variation of Rayleigh's quotient:

$$\begin{aligned}\delta\mathcal{R}(\mathbf{w}) &= \frac{2(\delta\mathbf{w}^T \mathbf{K}\mathbf{w})(\mathbf{w}^T \mathbf{M}\mathbf{w}) - 2(\mathbf{w}^T \mathbf{K}\mathbf{w})(\delta\mathbf{w}^T \mathbf{M}\mathbf{w})}{(\mathbf{w}^T \mathbf{M}\mathbf{w})^2} \\ &= \frac{2 \delta\mathbf{w}^T}{\mathbf{w}^T \mathbf{M}\mathbf{w}} \left( \mathbf{K}\mathbf{w} - \frac{\mathbf{w}^T \mathbf{K}\mathbf{w}}{\mathbf{w}^T \mathbf{M}\mathbf{w}} \mathbf{M}\mathbf{w} \right) \\ &= \frac{2 \delta\mathbf{w}^T}{\mathbf{w}^T \mathbf{M}\mathbf{w}} (\mathbf{K} - \mathcal{R}(\mathbf{w})\mathbf{M})\mathbf{w}\end{aligned}$$

If  $\mathbf{w} = \mathbf{p}_i$  then

$$(\mathbf{K} - \mathcal{R}(\mathbf{w})\mathbf{M})\mathbf{w} = (\mathbf{K} - \lambda_i \mathbf{M})\mathbf{p}_i = 0$$

**Stationary Rayleigh's quotient in the vicinity of an modal shape !**

## Rayleigh principle

Modal expansion:

$$\mathbf{w} = z_1 \mathbf{p}_1 + z_2 \mathbf{p}_2 + \cdots + z_i \mathbf{p}_i + \cdots + z_n \mathbf{p}_n = \mathbf{P}\mathbf{z}$$

where

$\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_i, \dots, \mathbf{p}_n]$  modal matrix

$\mathbf{z} = [z_1, z_2, \dots, z_i, \dots, z_n]^T$  coefficient vector  $[0, 1]$

Then

$$\mathcal{R}(\mathbf{w} = \mathbf{P}\mathbf{z}) = \frac{\mathbf{z}^T \mathbf{P}^T \mathbf{K} \mathbf{P} \mathbf{z}}{\mathbf{z}^T \mathbf{P}^T \mathbf{M} \mathbf{P} \mathbf{z}} = \frac{\mathbf{z}^T \mathbf{\Lambda} \mathbf{z}}{\mathbf{z}^T \mathbf{z}} = \frac{\sum_{i=1}^n \lambda_i z_i^2}{\sum_{i=1}^n z_i^2}$$

If, without loss of generality, we impose  $\mathbf{w}^T \mathbf{M} \mathbf{w} = 1$ , then  $\mathbf{z}^T \mathbf{z} = \sum_{i=1}^n z_i^2 = 1$  and

$$\mathcal{R}(\mathbf{w} = \mathbf{P}\mathbf{z}) = \sum_{i=1}^n \lambda_i z_i^2$$

## Rayleigh principle (continued)

$$\begin{aligned}\mathcal{R}(\mathbf{w}) &= \sum_{i=1}^n \lambda_i z_i^2 = \lambda_j - \lambda_j + \sum_{i=1}^n \lambda_i z_i^2 = \lambda_j - \lambda_j \left( \sum_{i=1}^n z_i^2 \right) + \sum_{i=1}^n \lambda_i z_i^2 \\ &= \lambda_j + \sum_{i=1}^n (\lambda_i - \lambda_j) z_i^2\end{aligned}$$

With eigenvalues ordered as  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots \leq \lambda_n$ , we have:  $\lambda_i - \lambda_j \geq 0$ .

For  $j = 1$ :

$$\mathcal{R}(\mathbf{w}) = \lambda_1 + \sum_{i=1}^n (\lambda_i - \lambda_1) z_i^2 \geq \lambda_1$$

For  $j = n$ :

$$\mathcal{R}(\mathbf{w}) = \lambda_n + \sum_{i=1}^n (\lambda_i - \lambda_n) z_i^2 \leq \lambda_n$$

## Rayleigh quotient bounding theorem

- Courant-Fischer formulas:

$$\mathcal{R}(\mathbf{w}) \geq \lambda_1 = \min_{\mathbf{w}} \mathcal{R}(\mathbf{w}) \quad \mathcal{R}(\mathbf{w}) \leq \lambda_n = \max_{\mathbf{w}} \mathcal{R}(\mathbf{w})$$

- Bounding theorem:

$$\lambda_1 \leq \mathcal{R}(\mathbf{w}) \leq \lambda_n$$

## Modal frequencies and shapes search via Rayleigh quotient minimization

(1) Find  $\lambda_1$  by minimization:

$$\lambda_1 = \min_{\mathbf{w}} \mathcal{R}(\mathbf{w})$$

(2) Find  $\mathbf{p}_1$  by solving  $(\mathbf{K} - \lambda_1 \mathbf{M})\mathbf{p}_1 = 0$ .

(3) Find  $\lambda_2$  by minimization:

$$\lambda_2 = \min_{\mathbf{w}} [\mathcal{R}(\mathbf{w}); \mathbf{w}^T \mathbf{M} \mathbf{p}_1 = 0]$$

(4) Find  $\mathbf{p}_2$  by solving  $(\mathbf{K} - \lambda_2 \mathbf{M})\mathbf{p}_2 = 0$ .

⋮

(j) Find  $\lambda_j$  by minimization:

$$\lambda_j = \min_{\mathbf{w}} [\mathcal{R}(\mathbf{w}); \mathbf{w}^T \mathbf{M} \mathbf{p}_i = 0, i = 1, 2, \dots, j-1]$$

(j+1) Find  $\mathbf{p}_j$  by solving  $(\mathbf{K} - \lambda_j \mathbf{M})\mathbf{p}_j = 0$ .

Example: Rayleigh's quotient for  
longitudinal vibrations of a free-free bar

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# MATLAB example - longitudinal vibrations of free-free bar

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## Numerical modal extraction algorithms

### 2. Subspace iteration

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# Inverse iteration algorithm

**Goal:** compute the eigenvector associated with the smallest eigenvalue of the generalized eigenproblem  $\mathbf{K}\mathbf{p} = \lambda\mathbf{M}\mathbf{p}$

**Inputs:**

- $\mathbf{K}$ ,  $\mathbf{M}$ : stiffness and mass matrices
- $\mathbf{p}^{(0)}$ : initial guess vector (non-zero)
- $\sigma$ : spectral shift (optional)
- $\varepsilon$ : convergence tolerance

**Algorithm:**

- ① If  $\mathbf{K}$  is singular, use shift: set  $\mathbf{K}_\sigma = \mathbf{K} + \sigma\mathbf{M}$
- ② For  $k = 1, 2, \dots$  until convergence:
  - Solve  $\mathbf{K}_\sigma \mathbf{p}^{(k)} = \mathbf{M} \mathbf{p}^{(k-1)}$
  - Normalize:  $\mathbf{p}^{(k)} \leftarrow \mathbf{p}^{(k)} / \|\mathbf{p}^{(k)}\|$
  - Check convergence:  $\|\mathbf{p}^{(k)} - \mathbf{p}^{(k-1)}\| < \varepsilon$

**Output:** approximated eigenvector  $\mathbf{p}^{(k)}$

# Subspace iteration method

**Goal:** compute the first  $m \ll n$  eigenpairs  $(\mathbf{p}_i, \lambda_i)$  of the generalized eigenproblem.

**Inputs:**

- $\mathbf{K}, \mathbf{M}$ : stiffness and mass matrices
- $\mathbf{P}^{(0)} \in \mathbb{R}^{n \times q}$ : initial guess (matrix with  $q > m$  linearly independent vectors)
- $\sigma$ : spectral shift (optional)
- $\varepsilon$ : convergence tolerance

**Output:**

- Approximated eigenvectors:  $\mathbf{P}^{(k)} = [\mathbf{p}_1^{(k)}, \dots, \mathbf{p}_q^{(k)}]$
- Approximated eigenvalues:  $\Lambda^{(k)} = \text{diag}(\lambda_1^{(k)}, \dots, \lambda_q^{(k)})$

**Algorithm:**

- ① If  $\mathbf{K}$  is singular, use shift: set  $\mathbf{K}_\sigma = \mathbf{K} + \sigma \mathbf{M}$
- ② For  $k = 1, 2, \dots$  until convergence:
  - Do steps 1, 2a, 2b and 2c
  - Check convergence

## Subspace iteration steps

- ① **Step 1:** Simultaneous inverse iteration on  $q > m$  vectors: fund the  $(n \times q)$  matrix  $\overline{\mathbf{P}^{(k)}}$  such that

$$\mathbf{K}\overline{\mathbf{P}^{(k)}} = \mathbf{M}\mathbf{P}^{(k-1)}$$

- ② **Step 2a:** Compute projected stiffness and mass matrices:

$$\mathbf{K}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{K} \overline{\mathbf{P}^{(k)}}, \quad \mathbf{M}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{M} \overline{\mathbf{P}^{(k)}}$$

- ③ **Step 2b:** Solve  $(q \times q)$  generalized eigenvalue problem: Find the modal matrix and the spectral matrix such that

$$\mathbf{K}^{(k)} \mathbf{Z}^{(k)} = \mathbf{M}^{(k)} \mathbf{Z}^{(k)} \Lambda^{(k)}$$

- ④ **Step 2c:** Orthogonalization:

$$\mathbf{P}^{(k)} = \overline{\mathbf{P}^{(k)}} \mathbf{Z}^{(k)}$$

## Subspace algorithm - step 1

Suppose that the Step 1 is replaced by a simultaneous inverse iteration on  $m$  eigenvectors:

$$\mathbf{P}^{(k)} = (\mathbf{K}^{-1}\mathbf{M})\mathbf{P}^{(k-1)} = \dots = (\mathbf{K}^{-1}\mathbf{M})^k \mathbf{P}_0$$

Define the subspace  $\mathcal{S}^{(k)}$  of rank  $q$ , spanned by the vectors  $\{\mathbf{p}_i^{(k)}\}$ .

$\mathbf{P}^{(k)} = [\mathbf{p}_1^{(k)}, \dots, \mathbf{p}_q^{(k)}]$  forms a *non-orthogonal* basis of  $\mathcal{S}^{(k)}$ .

- ✗ All columns of  $\mathbf{P}^{(k)}$  tend toward  $\mathbf{p}_1$
- ✗ Collinearity if no orthogonalization is applied !
  
- **Orthogonalization** of vectors  $\mathbf{p}_i^{(k)}$  at each iteration
- Use, for instance, Gram-Schmidt method (*Note: this step is computationally expensive*)

## Subspace algorithm - step 2a

- Orthogonalization by minimization of the Rayleigh quotient:

$$\mathcal{R}(\mathbf{w}^{(k)}) = \frac{(\mathbf{w}^{(k)})^T \mathbf{K} \mathbf{w}^{(k)}}{(\mathbf{w}^{(k)})^T \mathbf{M} \mathbf{w}^{(k)}}$$

- Let  $\mathbf{w}^{(k)} = \overline{\mathbf{P}^{(k)}} \mathbf{z}^{(k)}$
- Projected Rayleigh's quotient:

$$\mathcal{R}(\mathbf{w}^{(k)}) = \frac{(\mathbf{z}^{(k)})^T \mathbf{K}^{(k)} \mathbf{z}^{(k)}}{(\mathbf{z}^{(k)})^T \mathbf{M}^{(k)} \mathbf{z}^{(k)}}$$

where

$$\mathbf{K}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{K} \overline{\mathbf{P}^{(k)}}, \quad \mathbf{M}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{M} \overline{\mathbf{P}^{(k)}}$$

## Subspace algorithm - step 2b

- Minimization of the Projected Rayleigh's quotient (generalized eigenvalue problem of dimension  $q \times q$ )
- Stationary condition:

$$\delta \mathcal{R}(\mathbf{w}^{(k)}) = 0 \quad \Rightarrow \quad \mathbf{K}^{(k)} \mathbf{z}^{(k)} = \lambda^{(k)} \mathbf{M}^{(k)} \mathbf{z}^{(k)}$$

- Solve via transformation method (e.g., Jacobi method):

$$\mathbf{K}^{(k)} \mathbf{Z}^{(k)} = \mathbf{M}^{(k)} \mathbf{Z}^{(k)} \boldsymbol{\Lambda}^{(k)}$$

- Ritz vectors and values:

$$\mathbf{Z}^{(k)} = [\mathbf{z}_1^{(k)}, \dots, \mathbf{z}_q^{(k)}], \quad \text{and} \quad \boldsymbol{\Lambda}^{(k)} = \text{diag}(\lambda_1^{(k)}, \dots, \lambda_q^{(k)})$$

## Subspace algorithm - step 2c

- Update the modal matrix:

$$\mathbf{P}^{(k)} = \overline{\mathbf{P}^{(k)}} \mathbf{Z}^{(k)}$$

- Orthogonality check:

$$(\mathbf{P}^{(k)})^T \mathbf{M} \mathbf{P}^{(k)} = \mathbf{I}$$