Problem set 9

Problem 1

Use the Rayleigh minimization principle to determine the natural frequencies of a free-free shaft in torsion, assuming the structure is discretized using a single linear finite element. The strong form is given by

$$\frac{\partial}{\partial x}\left(GI_p\frac{\partial}{\partial x}\varphi(x,t)\right)=\rho I_p\ddot{\varphi}(x,t)$$

where the shaft parameters are:

- length ℓ,
- polar moment of inertia I_p,
- shear modulus G,
- mass density ρ.

The unknown φ represent the torsional vibration: angular displacement (twist angle) at position $x\in[0,\ell]$ and time $t\in[0,T]$. At $x=0$ and $x=\ell$ there are no external applied moments, this translates into natural boundary conditions:

$$\frac{\partial}{\partial x}\varphi(0,t)=\frac{\partial}{\partial x}\varphi(\ell,t)=0 \qquad \forall t\in]0,T[.$$

Problem 2

Develop a `MATLAB` script to compute the first two natural frequencies and the corresponding mode shapes associated with the longitudinal vibrations of a uniform free-free bar, using the subspace iteration method. The structure is characterized by the following physical parameters:

- length: $\ell=0.5$ m,
- cross-sectional area: $A=4\,10^{-4}$ m^2,
- Young's modulus: $E=210$ GPa,
- mass density: $\rho=7850$ kg/m^3.

The bar is modeled using a single quadratic finite element with three equally spaced nodes. The structure stiffness and lumped mass[1] matrices are given by:

$$\mathbf{K}=\frac{EA}{3\ell}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix} \qquad \text{and} \qquad \mathbf{M}=\frac{\rho A\ell}{6}\mathrm{diag}(1,4,1).$$

1) Formulate the generalized eigenvalue problem for the system, and define the stiffness and mass matrices explicitly.

[1]The use of a lumped mass matrix is often justified by its ability to simplify computations, improve numerical stability in explicit time integration schemes, and avoid non-physical oscillations associated with consistent mass formulations in low-order elements.

2) Since the system exhibits a rigid-body mode with zero frequency, a spectral shift is applied to ensure convergence. The modified eigenvalue problem is:

$$(\mathbf{K} + \sigma\mathbf{M})\mathbf{p} = (\lambda + \sigma)\mathbf{M}\mathbf{p}$$

where the shift parameter is chosen (to simplify the computation) as:

$$\sigma = 2\frac{E}{\rho\ell^2}.$$

Define the shifted stiffness matrix as $\mathbf{K}_\sigma = \mathbf{K} + \sigma\mathbf{M}$, yielding the equivalent eigenvalue problem:

$$\mathbf{K}_\sigma\mathbf{p} = \lambda_\sigma\mathbf{M}\mathbf{p}, \quad \text{with} \quad \lambda_\sigma = \lambda + \sigma.$$

3) Implement three iterations of the subspace iteration algorithm to approximate the two lowest eigenpairs $(\lambda_i, \mathbf{p}_i)$. For simplicity use the function `eig` to solve the reduced Rayleigh's minimization problem. Use the following initial guess:

$$\mathbf{P}_{(0)} = \frac{1}{\sqrt{\rho A\ell}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

4) Compare the numerical approximation with the exact (shifted) eigenvalues and corresponding eigenvectors (defined up to a multiplicative constant):

$$\lambda_{\sigma,1} = 2\frac{E}{\rho\ell^2} \qquad \text{and} \qquad \lambda_{\sigma,2} = 14\frac{E}{\rho\ell^2},$$

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and evaluate the accuracy by computing:

- The relative error between approximated and exact eigenvalues.
- The error in the **M**-norm between approximated and exact eigenvectors.