Dynamic analysis of Reissner-Mindlin plates

Classical structural elements

ME473 Dynamic finite element analysis of structures

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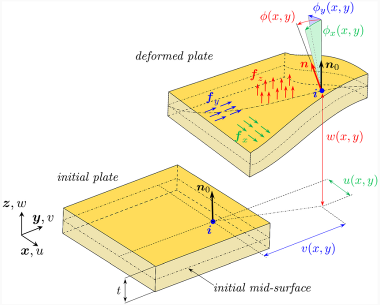
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation
3		FEM global	Project 1 statement
4		FEM local	
5		FEM local	Project 1 submission
6	Classical structural elements	Bars and trusses	Project 2 statement
7		Beams	
8		Frames and grids	
9		Kirchhoff-Love plates	Project 2 submission
10		Kirchhoff-Love plates	Project 3 statement
11		Reissner-Mindlin plates	

Summary

- Recap week 10
- Reissner-Mindlin plate theory
- Thick plate bending elements
- Example: modal analysis of a simply supported thick plate

Recommended readings

(N) Neto et al., Engineering Computation of Structures (chap. 6)

(P) Petyt, Introduction to finite element vibration analysis (chap. 6)

(O) Ochsner, PDE for classical structural members (chap. 7)

Recap week 10

Finite element approximation of Kirchhoff-Love plate

- Strong form

$$\frac{h^3}{12}\boldsymbol{\nabla}_k^T \boldsymbol{C}\boldsymbol{\nabla}_k u_3 + \rho h \ddot{u}_3 = f_3 \qquad \text{on } \Omega\times]0,T[$$

- Weak form equation

$$\frac{h^3}{12}\int_\Omega \boldsymbol{\nabla}_k u_3 \mathbf{C}\, \boldsymbol{\nabla}_k \delta u_3 \, d\Omega + \int_\Omega \rho h\, \ddot{u}_3\, \delta u_3 \, d\Omega = \int_\Omega f_3 \, \delta u_3 \, d\Omega$$

- Semi-discrete weak form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t)$$

- **Adini-Melosh-Clough element (AMC)**: 12 dofs quadrangular, not conforming, thin plate.
- **Crouzeix–Raviart (CR)**: 16 dofs quadrangular, conforming, thin plate.

Comparison: AMC vs CR plate bending element

CR element (Conforming)

- 4 degrees of freedom per node: u_3, θ_1, θ_2, and θ_{12}.
- Displacement u_3 and rotations θ_1, θ_2 are continuous across element boundaries.
- Fully conforming to the C^1 continuity required by Kirchhoff plate theory.
- Higher computational cost and complexity.

AMC element (Nonconforming)

- 3 degrees of freedom per node: u_3, θ_1, θ_2.
- Only displacement u_3 is continuous across elements; rotations may have jumps.
- Nonconforming element: does not fully satisfy C^1 continuity.
- Simpler and computationally cheaper; suitable for practical applications.

Selection of the displacement function

Displacement approximation for AMC element

$$
\begin{aligned}
{}^{e}u_3^h(x_1,x_2,t) = & \; a_1 + a_2x_1 + a_3x_2 + a_4x_1^2 + a_5x_1x_2 + a_6x_2^2 + \\
& + a_7x_1^3 + a_8x_1^2x_2 + a_9x_1x_2^2 + a_{10}x_2^3 + a_{11}x_1^3x_2 + a_{12}x_1x_2^3
\end{aligned}
$$

Displacement approximation for CR element

$$
\begin{aligned}
{}^{e}u_3^h(x_1,x_2,t) = & \; a_1 + a_2x_1 + a_3x_2 + a_4x_1^2 + a_5x_1x_2 + a_6x_2^2 + \\
& + a_7x_1^3 + a_8x_1^2x_2 + a_9x_1x_2^2 + a_{10}x_2^3 + \\
& + a_{11}x_1^3x_2 + a_{12}x_1x_2^3 + a_{13}x^2y^2 + a_{14}x^3y^2 + a_{15}x^2y^3 + a_{16}x^3y^3
\end{aligned}
$$

Approximate displacement and shape functions for the AMC element

The approximate displacement in local coordinate is defined as:

$$ {}^{e}u_3^h(\boldsymbol{\xi}, t) = \sum_{i=1}^{4} {}^{a}\mathbf{h}_i(\boldsymbol{\xi})\, {}^{e}\mathbf{q}^i(t) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\, {}^{e}\mathbf{q}(t) $$

- 3 dofs per node: ${}^{e}\mathbf{q}^i(t) = \begin{bmatrix} {}^{e}d^i(t) \\ {}^{e}\theta_1^i(t) \\ {}^{e}\theta_2^i(t) \end{bmatrix} = \begin{bmatrix} {}^{e}u_3^h(\boldsymbol{\xi}^i, t) \\ \partial_{\xi_2}\, {}^{e}u_3^h(\boldsymbol{\xi}^i, t)/b \\ -\partial_{\xi_1}\, {}^{e}u_3^h(\boldsymbol{\xi}^i, t)/a \end{bmatrix}$
- The shape function matrix is ${}^{a}\mathbf{H} = \begin{bmatrix} {}^{a}\mathbf{h}_1 & {}^{a}\mathbf{h}_2 & {}^{a}\mathbf{h}_3 & {}^{a}\mathbf{h}_4 \end{bmatrix}$ where

$$ {}^{a}\mathbf{h}_i(\boldsymbol{\xi}) = \begin{bmatrix} (1+\xi_1^i\xi_1)(1+\xi_2^i\xi_2)(2+\xi_1^i\xi_1+\xi_2^i\xi_2-\xi_1^2-\xi_2^2)/8 \\ b(1+\xi_1^i\xi_1)(\xi_2^i+\xi_2)(\xi_2^2-1)/8 \\ -a(\xi_1^i+\xi_1)(\xi_1^2-1)(1+\xi_2^i\xi_2)/8 \end{bmatrix}^T $$

Approximate displacement and shape functions for the CR element

The approximate displacement in local coordinate is defined as:

$$ {}^{e}u_3^h(\boldsymbol{\xi}, t) = \sum_{i=1}^{4} {}^{a}\mathbf{h}_i(\boldsymbol{\xi})\, {}^{e}\mathbf{q}^i(t) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\, {}^{e}\mathbf{q}(t) $$

- 4 dofs per node: $ {}^{e}\mathbf{q}^i(t) = \begin{bmatrix} {}^{e}d^i(t) \\ {}^{e}\theta_1^i(t) \\ {}^{e}\theta_2^i(t) \\ {}^{e}\theta_{12}^i(t) \end{bmatrix} = \begin{bmatrix} {}^{e}u_3^h(\boldsymbol{\xi}^i, t) \\ \partial_{\xi_2}\, {}^{e}u_3^h(\boldsymbol{\xi}^i, t)/b \\ -\partial_{\xi_1}\, {}^{e}u_3^h(\boldsymbol{\xi}^i, t)/a \\ \partial^2_{\xi_1\xi_2}\, {}^{e}u_3^h(\boldsymbol{\xi}^i, t)/(ab) \end{bmatrix} $

- The shape function matrix is ${}^{a}\mathbf{H} = \begin{bmatrix} {}^{a}\mathbf{h}_1 & {}^{a}\mathbf{h}_2 & {}^{a}\mathbf{h}_3 & {}^{a}\mathbf{h}_4 \end{bmatrix}$ where

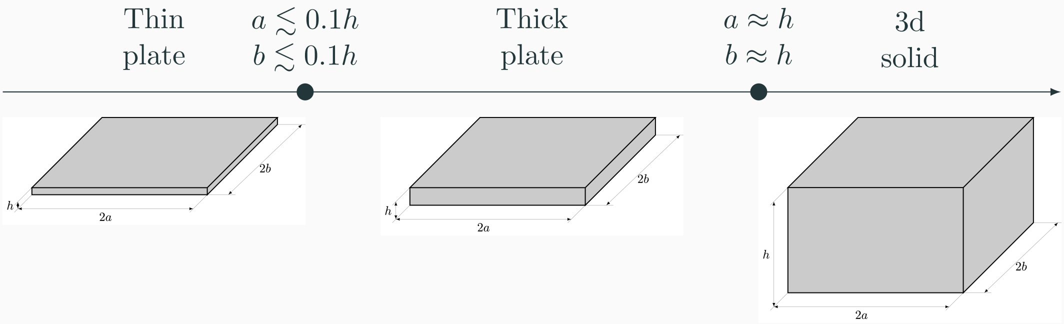
$$ {}^{a}\mathbf{h}_i(\boldsymbol{\xi}) = \begin{bmatrix} f_i(\xi_1) f_i(\xi_2) \\ b f_i(\xi_1) g_i(\xi_2) \\ -a g_i(\xi_1) f_i(\xi_2) \\ ab g_i(\xi_1) g_i(\xi_2) \end{bmatrix}^T $$

Hermite functions: $f_i(\xi) = (-\xi^i \xi^3 + 3\xi^i \xi + 2)/4$, $g_i(\xi) = (\xi^3 + \xi^i \xi^2 - \xi - \xi^i)/4$.

Reissner-Mindlin plate theory

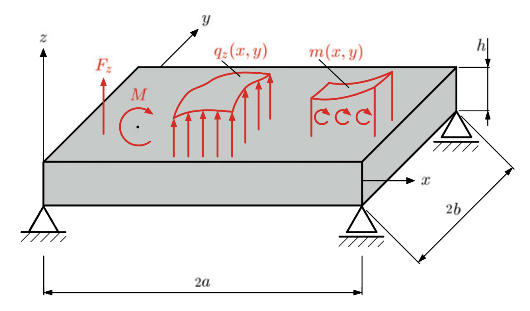
The validity of the classical (Kirchhoff-Love) plate theory depends on a number of factors:

1. the curvatures are small,
2. the in-plane plate dimensions are large compared to the thickness,
3. membrane strains are neglected.



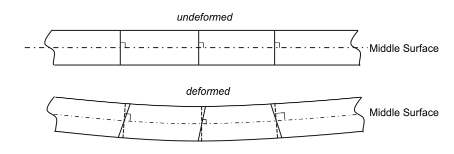
Shear deformable plates

- Thick plate are structures in which shear deformation and rotary inertia effects are important: transverse shear becomes an integral part of the formulation.
- The material is isotropic, homogenous and linear-elastic according to Hooke's law for a plane stress state where $\sigma_{33} = 0$,
- Plates carries only transversal loads and in-plane moments that lead to bending deformation of the plate.



Reissner-Mindlin assumption

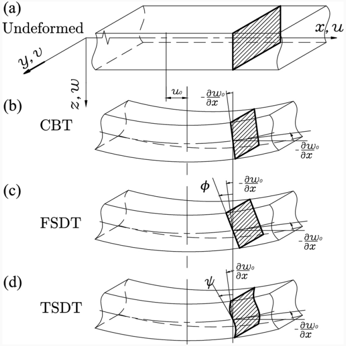
Rectilinearity of the cross-sectional normals: Timoshenko's hypothesis is valid, i.e. a cross-sectional plane stays planar and *but not necessarily perpendicular* to the middle surface in the deformed state.



Shear strains ε_{13} and ε_{23}, due to the distributed shear forces q_x and q_y, are constant through the thickness of the plate.

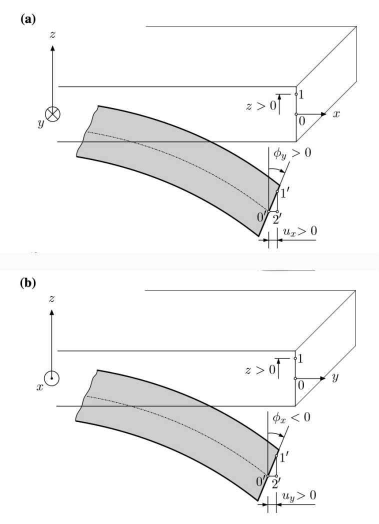
(Credit: (N))

Higher-order deformation theories



- Reissner-Mindlin theory is a First-Order Shear Deformation Theory.
- More advanced model: **Third-Order Plate Theory**.
  - Displacements varying as cubic functions through the thickness.
  - In-plane strains that are cubic in z.
  - Shear strains that are quadratic in z.
- In TSDT normal cross-sectional planes to the mid-surface can rotate and deform.

Kinematics assumptions



$$\phi_2 \approx \sin(\phi_2) = \frac{u_1}{z}$$

$$\implies u_1 = z\phi_2$$

$$\phi_1 \approx \sin(\phi_1) = -\frac{u_2}{z}$$

$$\implies u_2 = -z\phi_1$$

Independent variables:

- Transverse displacement u_3
- Rotation w.r.t. Ox axis ϕ_1
- Rotation w.r.t. Oy axis ϕ_2

Deformation is exaggerated in the figures for better illustration.

Strain-displacement relations

Using engineering definitions of strain: $\varepsilon_{ii} = \partial_i u_i$ and $\gamma_{ij} = \partial_i u_j + \partial_j u_i$ we obtain:

- **in-plane or bending strains**:

$$\underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}}_{\boldsymbol{\varepsilon_b}} = z \underbrace{\begin{bmatrix} 0 & 0 & \partial_x \\ 0 & -\partial_y & 0 \\ 0 & -\partial_x & \partial_y \end{bmatrix}}_{\nabla_b} \underbrace{\begin{bmatrix} u_3 \\ \phi_1 \\ \phi_2 \end{bmatrix}}_{\mathbf{u}},$$

- **transverse shear strains**:

$$\underbrace{\begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix}}_{\boldsymbol{\varepsilon_s}} = \underbrace{\begin{bmatrix} \partial_x & 0 & 1 \\ \partial_y & -1 & 0 \end{bmatrix}}_{\nabla_s} \underbrace{\begin{bmatrix} u_3 \\ \phi_1 \\ \phi_2 \end{bmatrix}}_{\mathbf{u}}.$$

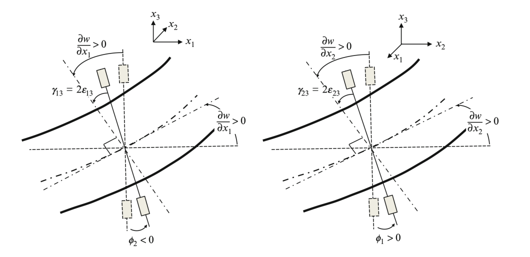
Note that $\varepsilon_{33} = 0$ due to the inextensibility of transverse fibers assumption.

Reduction to Kirchhoff-Love plate theory

The transverse shear are exactly equal to the additional rotations of the normal to the reference surface after deformation:

$$\gamma_{13} = \frac{\partial u_3}{\partial x} + \phi_2$$

$$\gamma_{23} = \frac{\partial u_3}{\partial y} - \phi_1$$



If the transverse shear strains are negligible, $\gamma_{13} = 0$ and $\gamma_{23} = 0$, then, as in the Kirchhoff-Love theory:

$$\frac{\partial u_3}{\partial x} = -\phi_2 \qquad \text{and} \qquad \frac{\partial u_3}{\partial y} = \phi_1$$

Constitutive equation for isotropic material

The constitutive equation for isotropic material is $\overline{\boldsymbol{\sigma}} = \overline{\mathbf{C}}\overline{\boldsymbol{\varepsilon}}$ where

- **bending stresses**:

$$\underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}}_{\boldsymbol{\sigma}_b} = \underbrace{\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{\mathbf{C}_b} \underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}}_{\boldsymbol{\varepsilon}_b},$$

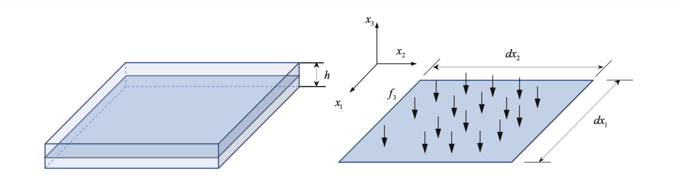
- **transverse shear stresses**:

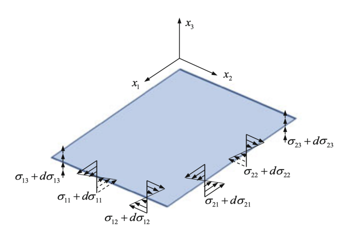
$$\underbrace{\begin{bmatrix} \sigma_{13} \\ \sigma_{23} \end{bmatrix}}_{\boldsymbol{\sigma}_s} = \underbrace{G \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{C}_s} \underbrace{\begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix}}_{\boldsymbol{\varepsilon}_s}.$$

Recall that, for isotropic materials, the shear modulus G is related to Young's modulus E and Poisson's ratio ν by: $G = \frac{E}{2(1+\nu)}$.

External forces and moments

Consider a plate cell of dimensions $dx_1 \times dx_2 \times h$ that is submitted to external forces, here denoted by f_3, and area distributed moments m_1 and m_2 (not shown).





Normal and shear stresses distributions through the thickness of the plate element:

- linear distributed normal stresses σ_{11} and σ_{22},
- linear distributed shear stresses σ_{12} and σ_{21},
- parabolic distributed transverse shear stresses σ_{13} and σ_{23}.

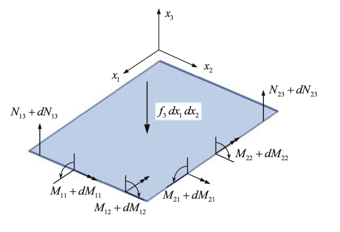
Normal and shear stresses distributions

- Transverse shear strains ε_{13} and ε_{23} are constant through the plate thickness (independent of z).
- Transverse shear stresses σ_{13} and σ_{23} also have a constant distribution through the plate thickness.
- *Transverse shear correction coefficient* k: accounts for the discrepancy in transverse shear stress between plate theory and 3D elasticity. It ensures the strain energy from shear stresses in plate theory matches that from 3D elasticity:

$$\underbrace{\begin{bmatrix}\sigma_{13}\\ \sigma_{23}\end{bmatrix}}_{\boldsymbol{\sigma}_s} = \underbrace{kG\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\mathbf{C}_s}\underbrace{\begin{bmatrix}\gamma_{13}\\ \gamma_{23}\end{bmatrix}}_{\boldsymbol{\varepsilon}_s}.$$

For homogeneous isotropic rectangular plates : $k = 5/6$.

Moments and shear forces



Moments and shear forces acting along the edge of the plate:

- bending moments M_{11} and M_{22},
- twisting moment M_{12},
- shear forces N_{13} and N_{23}.

$$\mathbf{M} = \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \, \boldsymbol{\sigma}_b \, dx_3 = \boldsymbol{C}_b \nabla_b \mathbf{u} \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3^2 \, dx_3 = \underbrace{\frac{h^3}{12} \boldsymbol{C}_b}_{\overline{\boldsymbol{C}_b}} \nabla_b \mathbf{u}$$

$$\mathbf{N} = \begin{bmatrix} N_{13} \\ N_{23} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{\sigma}_s \, dx_3 = \underbrace{h \boldsymbol{C}_s}_{\overline{\boldsymbol{C}_s}} \nabla_s \mathbf{u}$$

Moments and shear forces

In matrix form:

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{\boldsymbol{C}_b} & \mathbf{0} \\ \mathbf{0} & \overline{\boldsymbol{C}_s} \end{bmatrix}}_{\overline{\boldsymbol{C}}} \underbrace{\begin{bmatrix} \nabla_b \\ \nabla_s \end{bmatrix}}_{\nabla_r} \mathbf{u},$$

Here

$$\nabla_r = \begin{bmatrix} \nabla_b \\ \nabla_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & -\partial_y & 0 \\ 0 & -\partial_x & \partial_y \\ \partial_x & 0 & 1 \\ \partial_y & -1 & 0 \end{bmatrix},$$

$$\overline{\boldsymbol{C}_b} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \text{and} \quad \overline{\boldsymbol{C}_s} = \frac{Ekh}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Dynamic equilibrium equation

- Equilibrium condition for the vertical forces:

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + f_3 = \rho h \ddot{u}_3$$

- Equilibrium of moments:

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} + m_2 - N_{13} = \rho \frac{h^3}{12} \ddot{\phi}_2$$

$$\frac{\partial M_{22}}{\partial x_2} + \frac{\partial M_{12}}{\partial x_1} - m_1 - N_{23} = -\rho \frac{h^3}{12} \ddot{\phi}_1$$

Dynamic equilibrium equation

In matrix form:

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & \partial_x & \partial_y \\ 0 & -\partial_y & -\partial_x & 0 & 1 \\ \partial_x & 0 & \partial_y & -1 & 0 \end{bmatrix}}_{\nabla_m^T} \begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} + \underbrace{\begin{bmatrix} f_3 \\ m_1 \\ m_2 \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} \rho h & 0 & 0 \\ 0 & \rho h^3/12 & 0 \\ 0 & 0 & \rho h^3/12 \end{bmatrix}}_{\mathbf{I}} \ddot{\mathbf{u}}$$

- **I** mass moment of inertia matrix:
  - ρh translational inertia (in the transverse x_3-direction),
  - $\rho h^3/12$: rotational inertia (about the in-plane x_1- and x_2-axes).
- **f** applied forces and moments.
- Linear elastic stress-strain relation and the constitutive relation:

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} = \overline{\boldsymbol{C}} \nabla_r \mathbf{u}$$

Strong form for Reissner-Mindlin plate bending

Let $\Omega = [-a, a] \times [-b, b]$ be a rectangular plate. Find the transverse displacement $u_3 \in C^2(\Omega \times [0, T])$ and the rotations $\phi_1,\ \phi_2 \in C^2(\Omega \times [0, T])$ such that

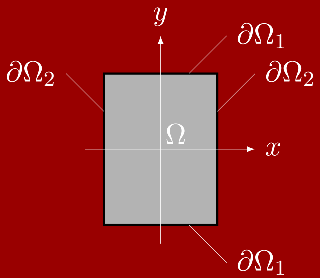
$$\nabla_m^T \overline{\boldsymbol{C}} \nabla_r \mathbf{u} + \mathbf{f} = \mathbf{I}\ddot{\mathbf{u}} \qquad \text{on } \Omega \times]0, T[$$

- boundary conditions (simply supported):

$$u_3 = 0 \qquad \text{in } \partial\Omega \times]0, T[$$
$$\phi_2 = 0 \qquad \text{in } \partial\Omega_1 \times]0, T[$$
$$\phi_1 = 0 \qquad \text{in } \partial\Omega_2 \times]0, T[$$

- initial conditions:

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in } \Omega$$
$$\dot{\mathbf{u}}(\cdot, 0) = \mathbf{v}_0 \quad \text{in } \Omega$$



Strong form for Reissner-Mindlin plate bending

Expanding the strong form, the three dynamic equations results in the following system of partial differential equations:

$$
\begin{bmatrix}
D_s\left(\partial^2_{xx} + \partial^2_{yy}\right) & -D_s\partial_y & D_s\partial_x \\
D_s\partial_y & D_b\left(\frac{1-\nu}{2}\partial^2_{xx} + \partial^2_{yy}\right) - D_s & -\frac{1+\nu}{2}D_b\partial^2_{xy} \\
-D_s\partial_x & -\frac{1+\nu}{2}D_b\partial^2_{xy} & D_b\left(\partial^2_{xx} + \frac{1-\nu}{2}\partial^2_{yy}\right) - D_s
\end{bmatrix}
\begin{bmatrix} u_3 \\ \phi_1 \\ \phi_2 \end{bmatrix} +
$$

$$
+ \begin{bmatrix} f_3 \\ m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \rho h\, \ddot{u}_3 \\ \rho h^3/12\, \ddot{\phi}_1 \\ \rho h^3/12\, \ddot{\phi}_2 \end{bmatrix}
$$

where $D_s = khG$, and $D_b = \frac{Eh^3}{12(1-\nu^2)}$.

Weak form for Reissner-Mindlin plate bending

The weak form consists of finding the transverse displacement $u_3 \in \mathcal{U}$ and the rotations $\phi_1,\ \phi_2 \in \mathcal{U}$ such that the following equation is satisfied for every $\boldsymbol{\delta}\mathbf{u} \in \mathcal{V}$:

$$\int_\Omega (\nabla_r \boldsymbol{\delta}\mathbf{u})^T\, \overline{\mathbf{C}}\, \nabla_r \mathbf{u}\, d\Omega + \int_\Omega \boldsymbol{\delta}\mathbf{u}^T\, \mathbf{I}\, \ddot{\mathbf{u}}\, d\Omega = \int_\Omega \boldsymbol{\delta}\mathbf{u}^T\, \mathbf{f}\, d\Omega$$

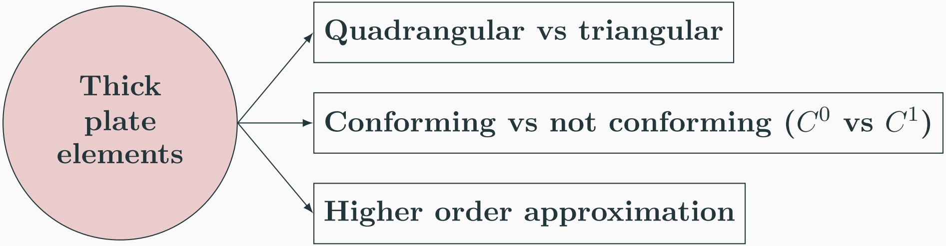
$$\mathcal{U} = \left\{ \mathbf{u}(\cdot, t) \in H^1(\Omega) \mid u_3(\cdot, t) = 0 \text{ in } \partial\Omega,\ \phi_1(\cdot, t) = 0 \text{ in } \partial\Omega_2,\ \phi_2(\cdot, t) = 0 \text{ in } \partial\Omega_1 \right\}$$

$$\mathcal{V} = \left\{ \boldsymbol{\delta}\mathbf{u} \in H^1(\Omega) \mid \delta u_3 = 0 \text{ in } \partial\Omega,\ \delta\phi_1 = 0 \text{ in } \partial\Omega_2,\ \delta\phi_2 = 0 \text{ in } \partial\Omega_1 \right\}$$

Thick plate bending elements

Overview of thick plate bending elements

Numerous finite elements for plate bending have been developed: **more than 88 distinct types** can be identified.



- **Hughes-Taylor-Kanoknukulcha element (HTK)**: 12 dofs quadrangular, not conforming, thick plate.

Generalized displacements approximation

Since the rotations ϕ_1 and ϕ_2 are defined independently of the transversal displacement u_3, the discretization procedure uses 2D bilinear finite elements.

$$ {}^{e}u_3(x,y,t) = \sum_{i=1}^{4} {}^{e}h_i(x,y)\, {}^{e}d^i(t) $$

$$ {}^{e}\phi_1(x,y,t) = \sum_{i=1}^{4} {}^{e}h_i(x,y)\, {}^{e}\theta_1^i(t) $$

$$ {}^{e}\phi_2(x,y,t) = \sum_{i=1}^{4} {}^{e}h_i(x,y)\, {}^{e}\theta_2^i(t) $$

Generalized displacements approximation

$$ {}^e\mathbf{u}^h(\mathbf{x},t) = {}^e\mathbf{H}(\mathbf{x})\,{}^e\mathbf{q}(t) = \sum_{i=1}^{4} {}^e h_i(\mathbf{x})\,{}^e\mathbf{q}^i(t) $$

- ${}^e\mathbf{H}(\mathbf{x})$ is a 3×12 matrix of **shape functions**:

$$ {}^e\mathbf{H} = \left[\; h_1\mathbf{I} \;\vdots\; h_2\mathbf{I} \;\vdots\; h_3\mathbf{I} \;\vdots\; h_4\mathbf{I} \;\right] = \begin{bmatrix} {}^e h_1 & 0 & 0 & & {}^e h_4 & 0 & 0 \\ 0 & {}^e h_1 & 0 & \dots & 0 & {}^e h_4 & 0 \\ 0 & 0 & {}^e h_1 & & 0 & 0 & {}^e h_4 \end{bmatrix} $$

  $\mathbf{I}$ is the 3×3 identity matrix.

- ${}^e\mathbf{q}^i(t) = \begin{bmatrix} {}^e d^i(t) \\ {}^e\theta_1^i(t) \\ {}^e\theta_2^i(t) \end{bmatrix}$ is the vector of generalized displacements of node i.

Elementary stiffness matrix

$$^e\mathbf{K} = \int_{^e\Omega} {^e\mathbf{B}}^T \overline{\mathbf{C}} {^e\mathbf{B}} \, d\Omega$$

- Elementary deformation matrix (5×12):

$$^e\mathbf{B} = \nabla_r {^e\mathbf{H}} = \left[\begin{array}{c:c:c} \nabla_r {^eh_1} & \ldots & \nabla_r {^eh_4} \end{array} \right] = \left[\begin{array}{c:c:c} \nabla_b {^eh_1} & \ldots & \nabla_b {^eh_4} \\ \nabla_s {^eh_1} & \ldots & \nabla_s {^eh_4} \end{array} \right] = \left[\begin{array}{c} \nabla_b {^e\mathbf{H}} \\ \nabla_s {^e\mathbf{H}} \end{array} \right]$$

- Bending strain-displacement matrix: $^e\mathbf{B}_b = \nabla_b {^e\mathbf{H}}$.
- Shear strain-displacement matrix: $^e\mathbf{B}_s = \nabla_s {^e\mathbf{H}}$.

- Constitutive matrix (5×5):

$$\overline{\boldsymbol{C}} = \begin{bmatrix} \overline{\boldsymbol{C}_b} & \mathbf{0} \\ \mathbf{0} & \overline{\boldsymbol{C}_s} \end{bmatrix}.$$

Elementary deformation matrix

- Bending strain-displacement matrix:

$$
{}^e\mathbf{B}_b = \nabla_b {}^e\mathbf{H} = \left[\begin{array}{ccc:c:ccc} 0 & 0 & \partial_x {}^e h_1 & \dots & 0 & 0 & \partial_x {}^e h_4 \\ 0 & -\partial_y {}^e h_1 & 0 & \dots & 0 & -\partial_y {}^e h_4 & 0 \\ 0 & -\partial_x {}^e h_1 & \partial_y {}^e h_1 & \dots & 0 & -\partial_x {}^e h_4 & \partial_y {}^a h_4 \end{array}\right].
$$

- Shear strain-displacement matrix:

$$
{}^e\mathbf{B}_s = \nabla_s {}^e\mathbf{H} = \left[\begin{array}{ccc:c:ccc} \partial_x {}^e h_1 & 0 & {}^e h_1 & \dots & \partial_x {}^e h_4 & 0 & {}^a h_4 \\ \partial_y {}^e h_1 & -{}^e h_1 & 0 & \dots & \partial_y {}^e h_4 & -{}^e h_4 & 0 \end{array}\right].
$$

Elementary stiffness matrix

The elementary stiffness matrix is split in two:

$$ {}^e\mathbf{K} = \int_{{}^e\Omega} \begin{bmatrix} {}^e\mathbf{B}_b \\ {}^e\mathbf{B}_s \end{bmatrix}^T \begin{bmatrix} \overline{\boldsymbol{C}_b} & \mathbf{0} \\ \mathbf{0} & \overline{\boldsymbol{C}_s} \end{bmatrix} \begin{bmatrix} {}^e\mathbf{B}_b \\ {}^e\mathbf{B}_s \end{bmatrix} d\Omega $$

$$ = \underbrace{\int_{{}^e\Omega} {}^e\mathbf{B}_b^T \overline{\boldsymbol{C}_b} {}^e\mathbf{B}_b \, d\Omega}_{{}^e\mathbf{K}_b} + \underbrace{\int_{{}^e\Omega} {}^e\mathbf{B}_s^T \overline{\boldsymbol{C}_s} {}^e\mathbf{B}_s \, d\Omega}_{{}^e\mathbf{K}_s} $$

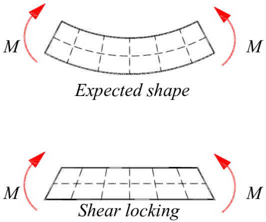
- bending stiffness matrix:

$$ {}^e\mathbf{K}_b = \int_{{}^e\Omega} {}^e\mathbf{B}_b^T \overline{\boldsymbol{C}_b} {}^e\mathbf{B}_b \, d\Omega, $$

- shear stiffness matrix:

$$ {}^e\mathbf{K}_s = \int_{{}^e\Omega} {}^e\mathbf{B}_s^T \overline{\boldsymbol{C}_s} {}^e\mathbf{B}_s \, d\Omega. $$

Selective integration to avoid shear locking



- Reissner-Mindlin theory has demonstrated to suffer from *shear locking*: as the thickness of the plate is reduced, the element becomes over-stiff and the computed displacements are much smaller than the analytical solution.
- The simplest remedy to this numerical behavior is to perform reduced integration of the shear component (selective integration).
- For instance, if bilinear elements are used, then: 2×2 Gauss integration (exact) for ${}^e\mathbf{K}_b$ and single point quadrature (reduced) for ${}^e\mathbf{K}_s$.

Elementary matrix and loads vector

- **Elementary mass matrix** (12×12):

$$ {}^{e}\mathbf{M} = \int_{{}^{e}\Omega} {}^{e}\mathbf{H}^{T}\, \mathbf{I}\, {}^{e}\mathbf{H}\, d\Omega. $$

- **Elementary applied forces vector** (12×1):

$$ {}^{e}\mathbf{r}(t) = \int_{{}^{e}\Omega} {}^{e}\mathbf{H}^{T} \mathbf{f}\, d\Omega. $$

Post processing: stress recovery

Once the nodal generalized displacements ${}^e\mathbf{q}^i$ is computed out stresses can be recovered from constitutive equations as:

$$\boldsymbol{\sigma}_b^h = \mathbf{C}_b \boldsymbol{\varepsilon}_b^h = z\mathbf{C}_b \nabla_b {}^e\mathbf{H}^e\mathbf{q} = z\mathbf{C}_b {}^e\mathbf{B}_b {}^e\mathbf{q},$$

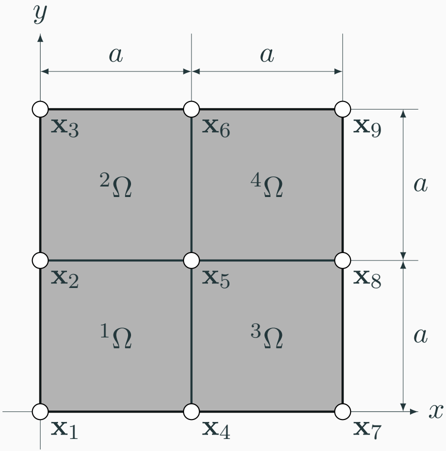
$$\boldsymbol{\sigma}_s^h = \mathbf{C}_s \boldsymbol{\varepsilon}_s^h = z\mathbf{C}_s \nabla_s {}^e\mathbf{H}^e\mathbf{q} = z\mathbf{C}_s {}^e\mathbf{B}_s {}^e\mathbf{q}.$$

- Since the bending stresses are linear through the plate thickness in the following they will be computed at the top layer of the plate $z = h/2$.
- On the contrary, shear stresses are constant through the thickness, thus they are independent on z.

Example: modal analysis of a simply supported thick plate

Example: Simply supported square isotropic plate

Discretization with 4 bilinear quadrilateral 2d elements (4 nodes each).



- $2a = 1$ length
- $2a = 1$ height
- $h = 0.1$ thickness
- $E = 10920$ Young's modulus
- $\nu = 0.3$ Poisson's ratio
- $\rho = 1$ material density
- $k = 5/6$ shear correction coefficient

The values for ρ and E is only a practical convenience to obtain non-dimensional flexural rigidity of the plate:

$$D = \frac{Eh^3}{12(1-\nu^2)} = 1.$$

Objectives

1. Approximate the first natural frequency of a $(1 \times 1 \times 0.1)$ plate, which is simply supported on all four edges. Use 4 bilinear quadrilateral 2d elements.

2. Compare the results with the analytical solution (as a function of h):

$$\omega_{1,1}^{\text{exact}}(h) = 20\pi^2 h \sqrt{\frac{70}{4\pi^2 h^2 + 7}} \text{ rad/s.}$$

Notice that this formula is only valid for the previous choice of the plate geometry and materials.

Modal patterns

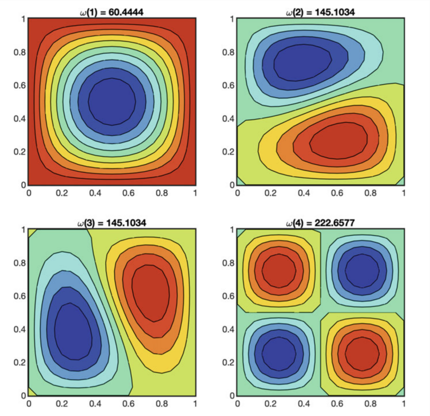


Figure 1: Modes of vibration for a SSSS plate with $h/a = 0.1$, using 20×20 bilinear elements.

(Credit: Ferreira, Fantuzzi - MATLAB Codes for Finite Element Analysis)

Step 1: Initialization and mesh generation

1. **Initialize variables**
   - Plate dimensions: `length_X`, `length_Y`, and `h`.
   - Material properties: `E`, `nu`, `rho`, and `kappa`.
   - Bending and shear stiffness matrices: `C_bending` and `C_shear`.
   - Inertia matrix: `Inertia_matrix`.
2. **Mesh generation**
   - Define number of elements in x and y directions: `number_elements_X` and `number_elements_Y`.
   - Compute total number of elements `number_elements`, nodes `number_nodes`, and DOFs `number_dofs`.
   - Generate structured rectangular mesh and build connectivity matrix: `createRectangularMesh()`.

Step 2: Shape functions and element matrices

3. **Bilinear shape functions**
   - Define $h_a(\xi_1, \xi_2)$: `ha(1)`,..., `ha(4)`.
4. **Transformations**
   - For each finite element `e` compute:
     - element transformation: $x(\xi), y(\xi)$: `transf{e}`.
     - jacobian matrix `jacobian_mat{e}`, its inverse `jacobian_inv{e}` and determinant `jacobian_det{e}`.
5. **Element matrices**
   - For each finite element `e` compute:
     - bending and shear strain-displacement matrices: `Be_bending{e}` and `Be_shear{e}`.
     - compute stiffness matrix: `K_elem{e} = K_elem_bending + K_elem_shear`.
     - Compute mass matrix `M_elem{e}`.

Step 3: Assembly of global matrices and boundary conditions

6 Assembly

- Initialize global matrices: stiffness and mass: `stiffness` and `mass`.
- For each element:
  - Map local to global DOFs.
  - Add contributions of local matrices to global matrices.

7 Boundary conditions

- Identify:
  - Corner nodes $\Rightarrow$ All DOFs fixed.
  - Edge nodes $\Rightarrow$ 2 DOFs fixed (displacement + one rotation).
- Build list of constrained DOFs: `constrained_local_dofs`.
- Derive list of free DOFs: `free_dofs`.

8 Solve the eigenvalue problem

- Reduce system matrices: `stiffness_freeDofs` and `mass_freeDofs`.
- Solve generalized eigenproblem and compute the fundamental frequency:

$$\omega_{1,1}^{\text{approx}} = \sqrt{\min(\lambda)}.$$

MATLAB example - simply supported plate

▸ Go to Matlab Drive