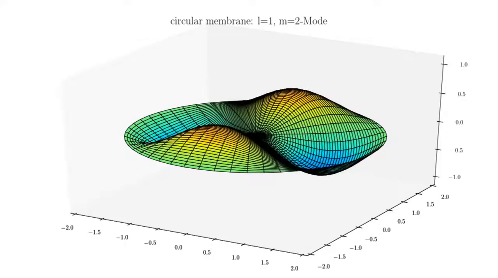
Transient analysis

Analysis of free and forced vibrations

ME473 Dynamic finite element analysis of structures

Stefano Burzio

2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	Groups formation
3		FEM global	Project 1 statement
4		FEM local	
5		FEM local	Project 1 submission
6	Classical structural elements	Bars and trusses	Project 2 statement
7		Beams	
8		Frames and grids	
9		Kirchhoff-Love plates	Project 2 submission
10		Kirchhoff-Love plates	Project 3 statement
11		Reissner-Mindlin plates	
12	Analysis of free and forced vibrations	Modal analysis methods	
13		Transient analysis	Project 3 submission

Summary

- General information
- Mini-project 3 comments
- Recap week 12
- Analysis of forced vibrations
- Direct integration methods

Recommended readings

(N) Neto et al., Engineering Computation of Structures (chap. 2.6)
(P) Petyt, Introduction to finite element vibration analysis (chap. 12)
(G) Gmür, Dynamique des structures (§5.1 and 5.2)

General end-of-course information

Final examination

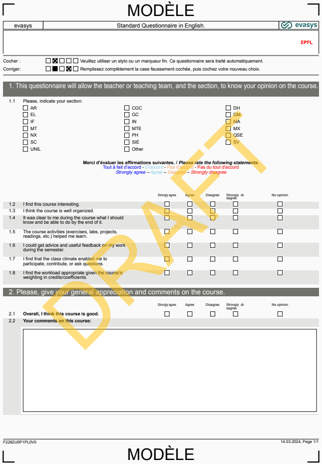
- **Date:** 26 June
- **Location:** Room CE1104
- **Duration:** 2 hours and 30 minutes
- **Permitted materials:** Open-book, electronic devices are not allowed (calculator excluded)

Avoid unnecessary stress!

Preparation recommendations:

- Thoroughly review all assigned problem sets.
- Do the mock exam, provided by 6 June.
- Attend the Q&A session: 18, 19, or 20 June.
- Use Ed-discussion forum or drop by my office (ME A2 390) if you have any questions.

CAPE evaluation survey



MODÈLE

evasys	Standard Questionnaire in English.	evasys

EPFL

Cocher : Veuillez utiliser un stylo ou un marqueur fin. Ce questionnaire sera traité automatiquement.
Corriger: Remplissez complètement la case faussement cochée, puis cochez votre nouveau choix.

1. This questionnaire will allow the teacher or teaching team, and the section, to know your opinion on the course.

1.1 Please, indicate your section:

☐ AR	☐ CGC	☐ DH
☐ EL	☐ GC	☐ GM
☐ IF	☐ IN	☐ MA
☐ MT	☐ MTE	☐ MX
☐ NX	☐ PH	☐ QSE
☐ SC	☐ SIE	☐ SV
☐ UNIL	☐ Other	

Merci d'évaluer les affirmations suivantes. / *Please rate the following statements.*
Tout à fait d'accord - D'accord - Pas d'accord - Pas du tout d'accord
Strongly agree – Agree – Disagree – Strongly disagree

		Strongly agree	Agree	Disagree	Strongly disagree	No opinion
1.2	I find this course interesting.	☐	☐	☐	☐	☐
1.3	I think the course is well organized.	☐	☐	☐	☐	☐
1.4	It was clear to me during the course what I should know and be able to do by the end of it.	☐	☐	☐	☐	☐
1.5	The course activities (exercises, labs, projects, readings, etc.) helped me learn.	☐	☐	☐	☐	☐
1.6	I could get advice and useful feedback on my work during the semester.	☐	☐	☐	☐	☐
1.7	I find that the class climate enabled me to participate, contribute, or ask questions.	☐	☐	☐	☐	☐
1.8	I find the workload appropriate given the course's weighting in credits/coefficients.	☐	☐	☐	☐	☐

2. Please, give your general appreciation and comments on the course.

		Strongly agree	Agree	Disagree	Strongly disagree	No opinion
2.1	**Overall, I think this course is good.**	☐	☐	☐	☐	☐

2.2 **Your comments on this course:**

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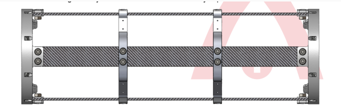
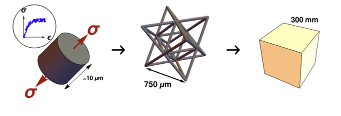
DRAFT

Please complete the in-depth evaluation survey!





Student assistant (AE) needed !

- The course will be offered in the Fall semester.
- We are seeking one, possibly two, teaching assistants (AEs) to support the course through:
  - Assistance with exercise sessions
  - Supervision of student mini-projects

Semester projects

- Structural assembly FE model validation via experimental modal analysis
- Neural networks meet finite elements

Master projects

- In collaboration with D-Orbit (topic to be finalized)

Mini-project 3 comments

A priori error estimates for eigenvalues and eigenvectors

Using principles from Rayleigh and Courant-Fischer, asymptotic error estimates can be established for eigenvalues and frequencies for **conformal elements**.

Error estimates:

$$\lambda_i \leq \lambda_i^h \leq \lambda_i + ch^{2(m-k+1)}\lambda_i^{m+1}$$

$$\omega_i \leq \omega_i^h \leq \omega_i + \bar{c}h^{2(m-k+1)}\omega_i^{2m+1}$$

- λ_i^h and ω_i^h are the approximated eigenvalues and frequencies
- λ_i and ω_i are the exact eigenvalues and frequencies,
- h represents the characteristic mesh size,
- m is the degree of the highest complete polynomial used,
- c and $\bar{c}$ are constants independent of h,
- k denotes the highest derivative order appearing in the weak form.

Abaqus file for mode shape visualization of thick and thin plates

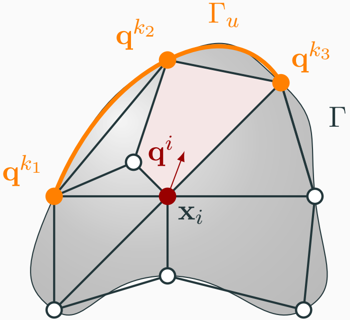
Go to Moodle week 11

Recap week 12

Free vibrations of non-rotating conservative systems

The discretization of linear three-dimensional elastodynamics, as well as the analysis of vibrations in beams and plates via FEM, all lead to a system of ODE:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t),$$



Free vibration: no external forcing is applied, i.e. $\mathbf{r}(t) = \mathbf{0}$.

- Generalized nodal displacements:

$$\mathbf{q}(t) = [\mathbf{q}^1(t), \ldots, \mathbf{q}^n(t)]^T.$$

- **Boundary conditions:** $\mathbf{q}^k = \hat{\mathbf{q}}^k$ for all k such that $\mathbf{x}_k \in \Gamma_u$.
- **Initial conditions:** $\mathbf{q}(0) = \mathbf{u}_0$ and $\dot{\mathbf{q}}(0) = \mathbf{v}_0$

Free undamped discrete vibration problem:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}$$

↓

Ansatz:

$$\mathbf{q}(t) = \alpha \mathbf{p} \cos(\omega t + \varphi)$$

↓

Generalized eigenvalue problem:

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{p} = 0$$

Solving the eigenvalue problem:

Eigenvalues (*natural frequencies* squared): $\lambda_j = \omega_j^2$ are the roots of the characteristic polynomial:

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0.$$

Eigenvectors (*modal shapes*): $\mathbf{p}_j$ are the solution of the equation

$$(\mathbf{K} - \lambda_j \mathbf{M})\mathbf{p}_j = \mathbf{0}.$$

Rigid body modes

In the semi-discrete weak form obtained via finite element discretization:

- The **mass matrix $\mathbf{M}$** is symmetric and strictly positive definite.
- The **stiffness matrix $\mathbf{K}$** is symmetric and positive semi-definite:

$$\mathbf{K}\mathbf{p} = \mathbf{0} \quad \text{for certain nonzero vectors } \mathbf{p}.$$

Consequently, the eigenvalues ω_j^2 of the generalized eigenvalue problem are all real and non-negative:

$$0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_n.$$

Rigid body modes: zero eigenvalues (i.e., $\omega_j = 0$) correspond to *rigid body motions*, where the system undergoes displacement without internal deformation.

Orthonormalization of mode shapes

Let $\mathbf{p}_i$ and $\mathbf{p}_j$ two eigenvectors corresponding to the eigenvalues λ_i and λ_j, then

$$\mathbf{p}_i^T \mathbf{M} \mathbf{p}_j = \delta_{ij} \qquad \text{and} \qquad \mathbf{p}_i^T \mathbf{K} \mathbf{p}_j = \omega_i^2 \delta_{ij}$$

where δ_{ij} represent Kronecker symbol.

Consequences: if we organize the modal vectors $\mathbf{p}_i$ in a so-called modal matrix $\mathbf{P}$:

$$\mathbf{P} = \left[\begin{array}{c:c:c:c} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \end{array} \right]$$

then

$$\mathbf{P}^T \mathbf{M} \mathbf{P} = \mathbf{I} \qquad \text{and} \qquad \mathbf{P}^T \mathbf{K} \mathbf{P} = \mathbf{\Lambda}$$

where $\mathbf{I}$ is the identity matrix of order n and $\mathbf{\Lambda}$ the spectral matrix:

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \mathrm{diag}(\omega_1^2, \dots, \omega_n^2).$$

Subspace iteration method

Goal: compute the first $m \ll n$ eigenpairs $(\mathbf{p}_i, \lambda_i)$ of the generalized eigenproblem.

Inputs:

- $\mathbf{K}$, $\mathbf{M}$: stiffness and mass matrices
- $\mathbf{P}^{(0)} \in \mathbb{R}^{n \times q}$: initial guess (matrix with $q > m$ linearly independent vectors)
- σ: spectral shift (optional)
- ε: convergence tolerance

Output:

- Approximated eigenvectors: $\mathbf{P}^{(k)} = [\mathbf{p}_1^{(k)}, \ldots, \mathbf{p}_q^{(k)}]$
- Approximated eigenvalues: $\boldsymbol{\Lambda}^{(k)} = \mathrm{diag}(\lambda_1^{(k)}, \ldots, \lambda_q^{(k)})$

Algorithm:

1. If $\mathbf{K}$ is singular, use shift: set $\mathbf{K}_\sigma = \mathbf{K} + \sigma\mathbf{M}$
2. For $k = 1, 2, \ldots$ until convergence:
    - Do steps 1, 2a, 2b and 2c
    - Check convergence

Subspace iteration steps

1. **Step 1:** Simultaneous inverse iteration on $q > m$ vectors: fund the $(n \times q)$ matrix $\overline{\mathbf{P}^{(k)}}$ such that
$$\mathbf{K}\overline{\mathbf{P}^{(k)}} = \mathbf{M}\mathbf{P}^{(k-1)}$$

2. **Step 2a:** Compute projected stiffness and mass matrices:
$$\mathbf{K}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T\mathbf{K}\overline{\mathbf{P}^{(k)}}, \quad \mathbf{M}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T\mathbf{M}\overline{\mathbf{P}^{(k)}}$$

3. **Step 2b:** Solve $(q \times q)$ generalized eigenvalue problem: Find the modal matrix and the spectral matrix such that
$$\mathbf{K}^{(k)}\mathbf{Z}^{(k)} = \mathbf{M}^{(k)}\mathbf{Z}^{(k)}\mathbf{\Lambda}^{(k)}$$

4. **Step 2c:** Orthogonalization:
$$\mathbf{P}^{(k)} = \overline{\mathbf{P}^{(k)}}\mathbf{Z}^{(k)}$$

Subspace algorithm - step 1

Suppose that the Step 1 is replace by a simultaneous inverse iteration on m eigenvectors:

$$\mathbf{P}^{(k)} = (\mathbf{K}^{-1}\mathbf{M})\mathbf{P}^{(k-1)} = \cdots = (\mathbf{K}^{-1}\mathbf{M})^k\mathbf{P}_0$$

Define the subspace $\mathcal{S}^{(k)}$ of rank q, spanned by the vectors $\{\mathbf{p}_i^{(k)}\}$.
$\mathbf{P}^{(k)} = [\mathbf{p}_1^{(k)}, \ldots, \mathbf{p}_q^{(k)}]$ forms a *non-orthogonal* basis of $\mathcal{S}^{(k)}$.

- ✗ All columns of $\mathbf{P}^{(k)}$ tend toward $\mathbf{p}_1$
- ✗ Collinearity if no orthogonalization is applied !

- **Orthogonalization** of vectors $\mathbf{p}_i^{(k)}$ at each iteration
- Use, for instance, Gram-Schmidt method (*Note: this step is computationally expensive*)

Subspace algorithm - step 2a

- Orthogonalization by minimization of the Rayleigh quotient:

$$\mathcal{R}(\mathbf{w}^{(k)}) = \frac{(\mathbf{w}^{(k)})^T\mathbf{K}\mathbf{w}^{(k)}}{(\mathbf{w}^{(k)})^T\mathbf{M}\mathbf{w}^{(k)}}$$

- Let $\mathbf{w}^{(k)} = \overline{\mathbf{P}^{(k)}}\mathbf{z}^{(k)}$
- Projected Rayleigh's quotient:

$$\mathcal{R}(\mathbf{w}^{(k)}) = \frac{(\mathbf{z}^{(k)})^T\mathbf{K}^{(k)}\mathbf{z}^{(k)}}{(\mathbf{z}^{(k)})^T\mathbf{M}^{(k)}\mathbf{z}^{(k)}}$$

where

$$\mathbf{K}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T\mathbf{K}\overline{\mathbf{P}^{(k)}}, \quad \mathbf{M}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T\mathbf{M}\overline{\mathbf{P}^{(k)}}$$

Subspace algorithm - step 2b

- Minimization of the Projected Rayleigh's quotient (generalized eigenvalue problem of dimension $q \times q$)
- Stationary condition:

$$\delta \mathcal{R}(\mathbf{w}^{(k)}) = 0 \qquad \Rightarrow \qquad \mathbf{K}^{(k)}\mathbf{z}^{(k)} = \lambda^{(k)}\mathbf{M}^{(k)}\mathbf{z}^{(k)}$$

- Solve via transformation method (e.g., Jacobi method):

$$\mathbf{K}^{(k)}\mathbf{Z}^{(k)} = \mathbf{M}^{(k)}\mathbf{Z}^{(k)}\mathbf{\Lambda}^{(k)}$$

- Ritz vectors and values:

$$\mathbf{Z}^{(k)} = [\mathbf{z}_1^{(k)}, \ldots, \mathbf{z}_q^{(k)}], \quad \text{and} \quad \mathbf{\Lambda}^{(k)} = \text{diag}(\lambda_1^{(k)}, \ldots, \lambda_q^{(k)})$$

Subspace algorithm - step 2c

- Update the modal matrix:

$$\mathbf{P}^{(k)} = \overline{\mathbf{P}^{(k)}}\mathbf{Z}^{(k)}$$

- Orthogonality check:

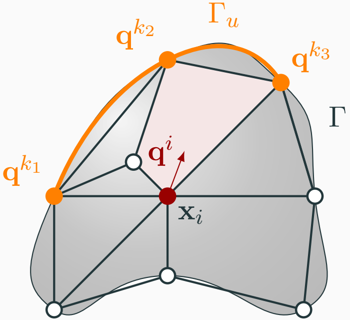
$$(\mathbf{P}^{(k)})^T\mathbf{M}\mathbf{P}^{(k)} = \mathbf{I}$$

Analysis of forced vibrations

Forced vibrations of non-rotating conservative systems

The discretization of linear three-dimensional elastodynamics, as well as the analysis of vibrations in beams and plates via FEM, all lead to a system of ODE:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t), \qquad \forall t \in [0, T]$$



- Generalized nodal displacements:

$$\mathbf{q}(t) = [\mathbf{q}^1(t), \ldots, \mathbf{q}^n(t)]^T.$$

- Excitation (force) vector: $\mathbf{r}(t) \neq \mathbf{0}$
- **Boundary conditions:** $\mathbf{q}^k = \hat{\mathbf{q}}^k$ for all k such that $\mathbf{x}_k \in \Gamma_u$.
- **Initial conditions:** $\mathbf{q}(0) = \mathbf{u}_0$ and $\dot{\mathbf{q}}(0) = \mathbf{v}_0$

Interest in finding the **temporal response** $\mathbf{q}(t)$ of the structure.

Modal basis

$$\mathbf{q}(t) = \mathbf{P}\mathbf{z}(t) = \sum_{i=1}^{n} \mathbf{p}_i z_i(t)$$

- $\mathbf{z}$: vector of modal coordinates
- $\mathbf{P}$: modal matrix
- $\mathbf{p}_i$: eigenvector of order i
- z_i: modal coordinate of order i

Insertion of the change of basis in the forced regime equation:

$$\mathbf{M}\mathbf{P}\ddot{\mathbf{z}}(t) + \mathbf{K}\mathbf{P}\mathbf{z}(t) = \mathbf{r}(t)$$

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P}\ddot{\mathbf{z}}(t) + \mathbf{P}^{\mathrm{T}}\mathbf{K}\mathbf{P}\mathbf{z}(t) = \mathbf{P}^{\mathrm{T}}\mathbf{r}(t)$$

Accounting for the orthogonality of eigenmodes

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{I} \qquad \text{and} \qquad \mathbf{P}^{\mathrm{T}}\mathbf{K}\mathbf{P} = \mathbf{\Lambda}$$

- $\mathbf{I}$: identity matrix
- $\mathbf{\Lambda} = \mathrm{diag}(\omega_1^2, \ldots, \omega_i^2, \ldots, \omega_n^2)$: diagonal matrix of eigenvalues,

Decoupling of the forced regime system:

$$\begin{aligned} \mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P}\ddot{\mathbf{z}}(t) + \mathbf{P}^{\mathrm{T}}\mathbf{K}\mathbf{P}\mathbf{z}(t) &= \mathbf{P}^{\mathrm{T}}\mathbf{r}(t) \\ \ddot{\mathbf{z}}(t) + \mathbf{\Lambda}\mathbf{z}(t) &= \mathbf{s}(t) \end{aligned}$$

- Projected initial conditions: $\mathbf{z}(0) = \mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{u}_0$ and $\dot{\mathbf{z}}(0) = \mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{v}_0$.
- $\mathbf{s}(t) = \mathbf{P}^{\mathrm{T}}\mathbf{r}(t)$: projection of $\mathbf{r}(t)$ onto the modal basis $\mathbf{P}$.

Exact solution

Component-wise form of the decoupled forced regime: $(i = 1, 2, \ldots, n)$

$$\begin{aligned} \ddot{z}_i(t) + \omega_i^2 z_i(t) &= s_i(t) \\ z_i(0) &= \mathbf{p}_i^{\mathrm{T}} \mathbf{M} \mathbf{u}_0 \\ \dot{z}_i(0) &= \mathbf{p}_i^{\mathrm{T}} \mathbf{M} \mathbf{v}_0 \end{aligned}$$

Exact solution by Laplace transform (convolution or Duhamel's integral):

$$\begin{aligned} \mathbf{q}(t) = &\sum_{i=1}^{n} \mathbf{p}_i \left(\frac{1}{\omega_i} \int_0^t s_i(t-\tau) \sin(\omega_i \tau) \, \mathrm{d}\tau \right) \\ &+ \sum_{i=1}^{n} \mathbf{p}_i \left(\mathbf{p}_i^{\mathrm{T}} \mathbf{M} \mathbf{u}_0 \cos(\omega_i t) + \frac{1}{\omega_i} \mathbf{p}_i^{\mathrm{T}} \mathbf{M} \mathbf{v}_0 \sin(\omega_i t) \right) \end{aligned}$$

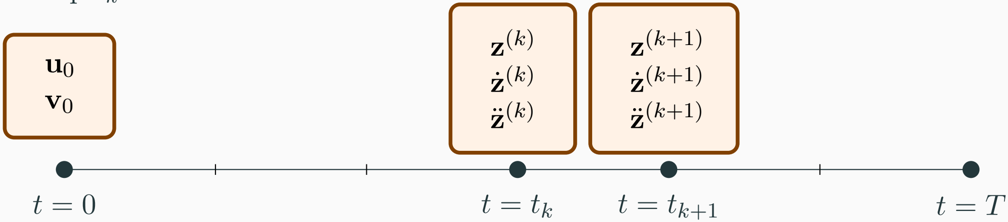
Direct integration methods

Finite difference method for time integration

- Solve dynamic equilibrium equations by replacing time derivatives with discrete approximations.
- Time domain $[0, T]$ is discretized into N_t equal intervals $\Delta t = T/N_t$:

$$t_k = k\Delta t \qquad k = 0, \ldots N_t.$$

- Displacement, velocity, and acceleration are approximated within each time step t_k.



Approximate solution via the finite difference method

General kinematic scheme (for second-order differential equation):

$$\begin{bmatrix} z_i^{(k)} \\ \dot{z}_i^{(k)} \end{bmatrix} = \sum_{j=1}^{q} \alpha_j \begin{bmatrix} z_i^{(k-j)} \\ \dot{z}_i^{(k-j)} \end{bmatrix} + \Delta t \sum_{j=0}^{q} \beta_j \begin{bmatrix} \dot{z}_i^{(k-j)} \\ \ddot{z}_i^{(k-j)} \end{bmatrix}, \quad k = 1, 2, \ldots; \quad k \geq q$$

- $z_i^{(k)}, \dot{z}_i^{(k)}, \ddot{z}_i^{(k)}$: component i of displacement, velocity, and acceleration at time step k,
- q, α_j, β_j for $j = 1, \ldots, q$: given constants,
- $\Delta t = t_k - t_{k-1}$: time step,
- k: time step index.

Types of time integration schemes

Single-step methods ($q = 1$):

- Use only the state at the previous time step to compute the current time step.
- Widely used due to balance between accuracy and efficiency.
- *Example:* Newmark methods are commonly used in finite element software.

Multi-step methods ($q > 1$):

- Use multiple previous time steps to compute the current time step.
- *Examples:* Houbolt's method, Wilson-θ method, Park's algorithm,

Explicit vs Implicit (β_0):

- Explicit: $\beta_0 = 0$, current values are computed directly from known past values.
- Implicit: $\beta_0 \neq 0$, require solving equations at each time step.

Classification of finite difference methods

General kinematic scheme:

$$\begin{bmatrix} z_i^{(k)} \\ \dot{z}_i^{(k)} \end{bmatrix} = \sum_{j=1}^{q} \alpha_j \begin{bmatrix} z_i^{(k-j)} \\ \dot{z}_i^{(k-j)} \end{bmatrix} + \Delta t \sum_{j=0}^{q} \beta_j \begin{bmatrix} \dot{z}_i^{(k-j)} \\ \ddot{z}_i^{(k-j)} \end{bmatrix}, \quad k = 1, 2, \ldots; \quad k \geq q$$

- $q = 1$: one-step scheme $\rightarrow$ Newmark method(s)
  - $\beta_0 = 0$: *explicit scheme* $\quad z_i^{(k)} \leftarrow z_i^{(k-1)}, \dot{z}_i^{(k-1)}$
  - $\beta_0 \neq 0$: *implicit scheme* $\quad z_i^{(k)} \leftarrow z_i^{(k-1)}, \dot{z}_i^{(k-1)}, \ddot{z}_i^{(k)}$
- $q > 1$: multi-step scheme $\rightarrow$ Park, Houbolt, Wilson, ...
  - $\beta_0 = 0$: *explicit scheme*
  - $\beta_0 \neq 0$: *implicit scheme*

Newmark's one-step method ($t = 0$)

Resolution of dynamic equations using a one-step Newmark's methods:

- Initial conditions:

$$z_i^{(0)} = \mathbf{p}_i^{\mathrm{T}} \mathbf{M} \mathbf{u}_0$$
$$\dot{z}_i^{(0)} = \mathbf{p}_i^{\mathrm{T}} \mathbf{M} \mathbf{v}_0$$

- Dynamic equation:

$$\ddot{z}_i^{(k)} + \omega_i^2 z_i^{(k)} = s_i^{(k)}$$

Computation of initial acceleration:

$$\ddot{z}_i^{(0)} = -\omega_i^2 z_i^{(0)} + s_i^{(0)}$$

Newmark's one-step method ($t = t_{k-1}$ to $t = t_k$)

Kinematic scheme for one-step Newmark's methods:

$$\begin{bmatrix} z_i^{(k)} \\ \dot{z}_i^{(k)} \end{bmatrix} = \alpha_1 \begin{bmatrix} z_i^{(k-1)} \\ \dot{z}_i^{(k-1)} \end{bmatrix} + \Delta t \left(\beta_0 \begin{bmatrix} \dot{z}_i^{(k)} \\ \ddot{z}_i^{(k)} \end{bmatrix} + \beta_1 \begin{bmatrix} \dot{z}_i^{(k-1)} \\ \ddot{z}_i^{(k-1)} \end{bmatrix} \right) \quad (k = 1, 2, \ldots)$$

- α_1, β_0, and β_1: parameters characterizing the different variants of the schemes.
- To compute α_1, β_0, and β_1 we use Taylor expansion with integral remainder:

$$\underbrace{z_i(t_k)}_{z_i^{(k)}} = \underbrace{z_i(t_{k-1})}_{z_i^{(k-1)}} + \Delta t \underbrace{\dot{z}_i(t_{k-1})}_{\dot{z}_i^{(k-1)}} + \int_{t_{k-1}}^{t_k} (t_k - \tau) \ddot{z}_i(\tau)\, d\tau$$

$$\underbrace{\dot{z}_i(t_k)}_{\dot{z}_i^{(k)}} = \underbrace{\dot{z}_i(t_{k-1})}_{\dot{z}_i^{(k-1)}} + \int_{t_{k-1}}^{t_k} \ddot{z}_i(\tau)\, d\tau$$

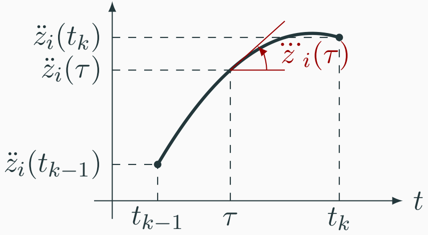
First-order integral remainder approximation

Taylor series for acceleration:
let $\beta \in]0, 1/2[$ a constant, then

$$\times(1-2\beta) \qquad \ddot{z}_i(t_{k-1}) = \ddot{z}_i(\tau) - (\tau - t_{k-1})\dddot{z}_i(\tau) + \dots$$

$$\times 2\beta \qquad \ddot{z}_i(t_k) = \ddot{z}_i(\tau) + (t_k - \tau)\dddot{z}_i(\tau) + \dots$$

$$\Rightarrow (1-2\beta)\ddot{z}_i(t_{k-1}) + 2\beta\ddot{z}_i(t_k) \approx \ddot{z}_i(\tau)$$



Notice that terms in $\dddot{z}_i(\tau)$ and higher are neglected because they are multiplied in displacement by Δt^3, and in velocity by Δt^2.

Taylor formula with integral remainder for modal displacements

Integral quadrature in Taylor expansion:

$$\int_{t_{k-1}}^{t_k} (t_k - \tau)\ddot{z}_i(\tau)\, d\tau = \int_{t_{k-1}}^{t_k} (t_k - \tau) \left[(1 - 2\beta)\ddot{z}_i^{(k-1)} + 2\beta\ddot{z}_i^{(k)}\right] d\tau$$

$$= \dots$$

$$= \Delta t^2 \left[\left(\frac{1}{2} - \beta\right)\ddot{z}_i^{(k-1)} + \beta\ddot{z}_i^{(k)}\right]$$

Kinematic scheme for displacements:

$$z_i^{(k)} = z_i^{(k-1)} + \Delta t \dot{z}_i^{(k-1)} + \Delta t^2 \left[\left(\frac{1}{2} - \beta\right)\ddot{z}_i^{(k-1)}\right] + \beta \Delta t^2 \ddot{z}_i^{(k)}$$

First-order integral remainder approximation

Taylor series for acceleration: let $\gamma \in]0,1[$ a second constant, then

$$\Rightarrow (1-\gamma)\ddot{z}_i(t_{k-1}) + \gamma \ddot{z}_i(t_k) \approx \ddot{z}_i(\tau)$$

Integral quadrature in Taylor expansion:

$$\int_{t_{k-1}}^{t_k} \ddot{z}_i(\tau)\, d\tau = \int_{t_{k-1}}^{t_k} \left[(1-\gamma)\ddot{z}_i^{(k-1)} + \gamma \ddot{z}_i^{(k)}\right] d\tau = \Delta t \left[(1-\gamma)\, \ddot{z}_i^{(k-1)} + \gamma \ddot{z}_i^{(k)}\right]$$

Kinematic scheme for velocity:

$$\dot{z}_i^{(k)} = \dot{z}_i^{(k-1)} + \Delta t \left[(1-\gamma)\, \ddot{z}_i^{(k-1)}\right] + \gamma \Delta t \ddot{z}_i^{(k)}$$

Newmark's one-step method ($t = t_{k-1}$ to $t = t_k$)

Stability control parameter β: $0 \leq \beta \leq 1/4$

Numerical dissipation parameter γ: $1/2 \leq \gamma \leq 3/4$.

- **Kinematic scheme for displacement:**

$$z_i^{(k)} = z_i^{(k-1)} + \Delta t \dot{z}_i^{(k-1)} + \Delta t^2 \left[\left(\frac{1}{2} - \beta \right) \ddot{z}_i^{(k-1)} \right] + \beta \Delta t^2 \ddot{z}_i^{(k)}$$

- **Kinematic scheme for velocity:**

$$\dot{z}_i^{(k)} = \dot{z}_i^{(k-1)} + \Delta t \left[(1 - \gamma) \ddot{z}_i^{(k-1)} \right] + \gamma \Delta t \ddot{z}_i^{(k)}$$

- **Dynamic equation:**

$$\ddot{z}_i^{(k)} + \omega_i^2 z_i^{(k)} = s_i^{(k)}$$

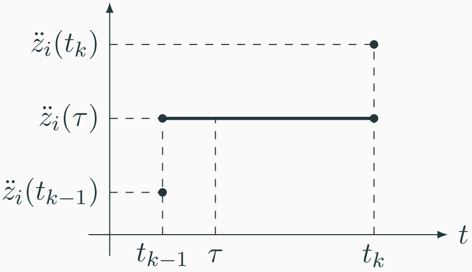
Average acceleration method (Trapezoidal rule)

Special case:

$$\beta = \frac{1}{4} \quad \text{and} \quad \gamma = \frac{1}{2}$$

Then for $t_{k-1} \leq \tau \leq t_k$ we have

$$\ddot{z}_i(\tau) = \frac{\ddot{z}_i(t_{k-1}) + \ddot{z}_i(t_k)}{2}$$



Kinematic schemes for displacement and velocity:

$$z_i^{(k)} = z_i^{(k-1)} + \Delta t \dot{z}_i^{(k-1)} + \frac{1}{4}\Delta t^2 \left(\ddot{z}_i^{(k-1)} + \ddot{z}_i^{(k)}\right)$$

$$\dot{z}_i^{(k)} = \dot{z}_i^{(k-1)} + \frac{1}{2}\Delta t \left(\ddot{z}_i^{(k-1)} + \ddot{z}_i^{(k)}\right)$$

"The person who learns the most in any classroom is the teacher."

— *James Clear*

Please complete the in-depth evaluation survey!