Problem set 10 - solutions

Problem 1

1. Unconstrained and constrained degrees of freedom. The bar is clamped at node 1, hence the corresponding displacement is constrained to zero: $d_1(t) = 0$. The unknown degrees of freedom are the displacements along the longitudinal axis at the unconstrained nodes 2 and 3, which are collected in the vector

$$\mathbf{q}_f(t) = \begin{bmatrix} d_2(t) \\ d_3(t) \end{bmatrix}$$

To proceed, the global stiffness and mass matrices are partitioned accordingly, yielding:

$$\mathbf{K}_f = \frac{EA}{3\ell}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}, \quad \mathbf{M}_f = \frac{\rho A \ell}{6}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{r}_f(t) = \begin{bmatrix} 0 \\ \delta(t) \end{bmatrix}.$$

The differential equations governing the forced response of the bar is then written, taking into account the essential boundary condition at the clamped end, as follows:

$$\frac{\rho A \ell}{6}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{d}_2(t) \\ \ddot{d}_3(t) \end{bmatrix} + \frac{EA}{3\ell}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} d_2(t) \\ d_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \delta(t) \end{bmatrix}$$

The structure is initially at rest, thus the initial conditions are:

$$d_2(0) = d_3(0) = 0,$$
$$\dot{d}_2(0) = \dot{d}_3(0) = 0.$$

2. Exact solution. To determine the dynamic behavior of the bar under the action of an external load, we first evaluate the solution of equation without the right-hand side (free response), which will then allow us to apply the modal superposition method using the modal parameters obtained in the first step.
We solve the generalized eigenvalue problem : $\mathbf{K}_f p_i = \lambda_i \mathbf{M}_f p_i$, for $i = 1, 2$. To simplify computations let:

$$\mathbf{K}_f = K_0\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}, \quad \mathbf{M}_f = M_0\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

with $K_0 = \frac{EA}{3\ell}$ and $M_0 = \frac{\rho A \ell}{6}$. We solve: $(\mathbf{K}_f - \lambda \mathbf{M}_f)\,\mathbf{p} = \mathbf{0}$. The characteristic polynomial is:

$$\det\left(\begin{bmatrix} 16K_0 - 4M_0\lambda & -8K_0 \\ -8K_0 & 7K_0 - M_0\lambda \end{bmatrix}\right) = 0$$

This simplifies to a quadratic in λ:

$$4M_0^2\lambda^2 - 44K_0M_0\lambda + 48K_0^2 = 0,$$

which reduces to:

$$\lambda^2 - 22\frac{E}{\rho\ell}\lambda + 48\frac{E^2}{\rho^2\ell^2} = 0.$$

Solving this yields the eigenvalues:

$$\lambda_1 = (11 - \sqrt{73})\frac{E}{\rho\ell^2} = 2.4560\frac{E}{\rho\ell^2} \qquad \text{and} \qquad \lambda_2 = (11 + \sqrt{73})\frac{E}{\rho\ell^2} = 19.5440\frac{E}{\rho\ell^2}.$$

and corresponding natural frequencies:

$$\omega_1 = 1.5672\sqrt{\frac{E}{\rho\ell^2}} \qquad \text{and} \qquad \omega_2 = 4.4209\sqrt{\frac{E}{\rho\ell^2}}.$$

Substituting the eigenvalues into the generalized eigenvalue problem allows us to extract the mode shapes $\mathbf{p}_1$ and $\mathbf{p}_2$, which we then normalize with respect to the diagonal mass matrix $\mathbf{M}_f$.

$$\mathbf{p}_1 = \frac{1}{\sqrt{\rho A\ell}}\begin{bmatrix}1.0066\\1.3952\end{bmatrix}, \quad \mathbf{p}_2 = \frac{1}{\sqrt{\rho A\ell}}\begin{bmatrix}-0.6976\\2.0133\end{bmatrix}$$

Let $\mathbf{P} = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{bmatrix}$ be the modal matrix of eigenvectors normalized with respect to $\mathbf{M}_f$:

$$\mathbf{P}^T\mathbf{M}_f\mathbf{P} = \mathbf{I}.$$

The modal parameters are now known. The modal superposition technique can be employed to determine the time response of the bar to a given excitation. We introduce the change of variables:

$$\mathbf{q}_f(t) = \mathbf{P}\mathbf{z}_f(t),$$

which transforms the governing system into a set of uncoupled equations:

$$\ddot{\mathbf{z}}_f(t) + \mathbf{\Lambda}\mathbf{z}_f(t) = \mathbf{P}^T\mathbf{r}_f(t)$$

where $\mathbf{\Lambda} = \text{diag}(\omega_1^2, \omega_2^2)$. Since $\mathbf{r}_f(t) = \begin{bmatrix}0\\\delta(t)\end{bmatrix}$, we have:

$$\mathbf{s}_f(t) = \mathbf{P}^T\mathbf{r}_f(t) = \begin{bmatrix}p_{12}\\p_{22}\end{bmatrix}\delta(t).$$

The system decouples into two second-order scalar differential equations: $(i = 1, 2)$:

$$\ddot{z}_i(t) + \omega_i^2 z_i(t) = p_{i2}\delta(t).$$

The exact solution, using Duhamel's integral or Laplace transforms, is:

$$\begin{aligned} z_i(t) &= \frac{1}{\omega_i}\int_0^t p_{i2}\delta(t-\tau)\sin(\omega_i\tau)\,\mathrm{d}\tau \\ &= \frac{1}{\omega_i}p_{i2}\sin(\omega_i t). \end{aligned}$$

Returning to the physical coordinates via $\mathbf{q}_f(t) = \mathbf{P}\mathbf{z}_f(t)$, we obtain the full dynamic response:

$$\mathbf{q}_f(t) = \mathbf{p}_1 z_1(t) + \mathbf{p}_2 z_2(t).$$

Explicitly, the displacements at nodes 2 and 3 are:

$$d_2(t) = \frac{p_{11}\,p_{12}}{\omega_1}\sin(\omega_1 t) + \frac{p_{21}\,p_{22}}{\omega_2}\sin(\omega_2 t) = \frac{1}{A\sqrt{\rho E}}\big(0.8958\sin(1.5672\,t) - 0.3178\sin(4.4209\ t)\big)$$
$$d_3(t) = \frac{(p_{12})^2}{\omega_1}\sin(\omega_1 t) + \frac{(p_{22})^2}{\omega_2}\sin(\omega_2 t) = \frac{1}{A\sqrt{\rho E}}\big(1.2431\sin(1.5672\,t) + 0.9168\sin(4.4209\,t)\big)$$

3. Average acceleration Newmark method. Recall from the previous section that the modal equations are:

$$\ddot{z}_1(t) + 2.4560\,\frac{E}{\rho A\ell^2}\,z_1(t) = 1.3952\,\frac{1}{\sqrt{\rho A\ell}}\,\delta(t),$$
$$\ddot{z}_2(t) + 19.544\,\frac{E}{\rho A\ell^2}\,z_2(t) = 2.0133\,\frac{1}{\sqrt{\rho A\ell}}\,\delta(t).$$

At $t = 0$, the bar is at rest $z_1(0) = \dot{z}_1(0) = 0$ and $z_2(0) = \dot{z}_2(0) = 0$, thus $z_1^{(0)} = \dot{z}_1^{(0)} = 0$ and $z_2^{(0)} = \dot{z}_2^{(0)} = 0$. Based on the geometric and material properties of the bar and the definition of the time excitation function, the problem reduces to step-by-step iteration using the following dynamic relations:

$$\begin{aligned}\ddot{z}_1^{(0)} + (8105.7)^2 z_1^{(0)} &= 15747,\\ \ddot{z}_2^{(0)} + (22866.2)^2 z_2^{(0)} &= 22723.\end{aligned}\tag{1}$$

and for every time step index $k \geq 1$:

$$\begin{aligned}\ddot{z}_1^{(k)} + (8105.7)^2 z_1^{(k)} &= 0,\\ \ddot{z}_2^{(k)} + (22866.2)^2 z_2^{(k)} &= 0.\end{aligned}\tag{2}$$

We begin by completing the initial phase of the Newmark's scheme: that is computing the accelerations at the initial time step using dynamic equations (1). Since the initial modal coordinates are zero, the initial modal accelerations become:

$$\ddot{z}_1^{(0)} = 15747 - (8105.7)^2 z_1^{(0)} = 15747$$
$$\ddot{z}_2^{(0)} = 22723 - (22866.2)^2 z_2^{(0)} = 22723$$

To compute the modal displacements, velocities, and accelerations at $t = \Delta t$, we employ the two kinematic relations associated with the Newmark average acceleration method, which for each $i = 1, 2$ take the form:

$$\begin{aligned} z_i^{(k)} &= \underbrace{z_i^{(k-1)} + \Delta t\,\dot{z}_i^{(k-1)} + 0.25\Delta t^2\ddot{z}_i^{(k-1)}}_{\hat{z}_i^{(k-1)}} + 0.25\Delta t^2\ddot{z}_i^{(k)}\\ \dot{z}_i^{(k)} &= \underbrace{\dot{z}_i^{(k-1)} + 0.5\Delta t\ddot{z}_i^{(k-1)}}_{\hat{\dot{z}}_i^{(k-1)}} + 0.5\Delta t\ddot{z}_i^{(k)}\end{aligned}\tag{3}$$

here $k = 1, 2, \dots$ denotes the time step index. It is clear that due to the implicit nature of the resolution method, the modal accelerations from the dynamic equations (2), can only be extracted after being inserted into the kinematic relations (3).
At the first iteration of the process, the prediction, obtained from the truncated kinematic relations (3) (displacements and modal velocities) using the values from the initial step, yields the following result:

$$\begin{aligned}
\hat{z}_1^{(1)} &= z_1^{(0)} + \Delta t\, \dot{z}_1^{(0)} + 0.25\Delta t^2 \ddot{z}_1^{(0)} = 0.25 \cdot (10^{-4})^2 \cdot 15747 = 0.39368 \cdot 10^{-4} \\
\hat{\dot{z}}_1^{(1)} &= \dot{z}_1^{(0)} + 0.5\Delta t \ddot{z}_1^{(0)} = 0.5 \cdot (10^{-4}) \cdot 15747 = 0.78737 \\
\hat{z}_2^{(1)} &= z_2^{(0)} + \Delta t\, \dot{z}_2^{(0)} + 0.25\Delta t^2 \ddot{z}_2^{(0)} = 0.25 \cdot (10^{-4})^2 \cdot 22723 = 0.56808 \cdot 10^{-4} \\
\hat{\dot{z}}_2^{(1)} &= \dot{z}_2^{(0)} + 0.5\Delta t \ddot{z}_2^{(0)} = 0.5 \cdot (10^{-4}) \cdot 22723 = 1.1362
\end{aligned}$$

This predictions $\hat{z}_i^{(1)}$ and $\hat{\dot{z}}_i^{(1)}$ are then substituted into the dynamic equations (2) and kinematic equations (3), to yield the following implicit relations:

$$\begin{aligned}
\ddot{z}_1^{(1)} &= -(8105.7)^2 z_1^{(1)} = -(8105.7)^2 (\hat{z}_1^{(1)} + 0.25\Delta t^2 \ddot{z}_1^{(1)}) \\
\ddot{z}_2^{(1)} &= -(22866.2)^2 z_2^{(1)} = -(22866.2)^2 (\hat{z}_2^{(1)} + 0.5\Delta t \ddot{z}_2^{(1)})
\end{aligned}$$

Solving for the accelerations gives:

$$\begin{aligned}
\ddot{z}_1^{(1)} &= \left[-(8105.7)^2 \hat{z}_1^{(1)}\right] / \left[1 + 0.25 \cdot (8105.7)^2 \cdot \Delta t^2\right] \\
&= \left[-(8105.7)^2 \cdot 0.39368 \cdot 10^{-4}\right] / \left[1 + 0.25 \cdot (8105.7)^2 \cdot (10^{-4})^2\right] \\
&= -2221.6516
\end{aligned}$$

$$\begin{aligned}
\ddot{z}_2^{(1)} &= \left[-(22866.2)^2 \hat{z}_2^{(1)}\right] / \left[1 + 0.25 \cdot (22866.2)^2 \cdot \Delta t^2\right] \\
&= \left[-(22866.2)^2 \cdot 0.56808 \cdot 10^{-4}\right] / \left[1 + 0.25 \cdot (22866.2)^2 \cdot (10^{-4})^2\right] \\
&= -12874.1985
\end{aligned}$$

It is then possible to correct the kinematic predictions $\hat{z}_i^{(1)}$ and $\hat{\dot{z}}_i^{(1)}$ by incorporating the above values to complete the equations and obtain the modal displacements and velocities at the first time step:

$$\begin{aligned}
z_1^{(1)} &= \hat{z}_1^{(1)} + 0.25\, \Delta t^2\, \ddot{z}_1^{(1)} = 0.39368 \cdot 10^{-4} + 0.25 \cdot (10^{-4})^2 \cdot (-2221.6516) = 3.3813 \cdot 10^{-5} \\
\dot{z}_1^{(1)} &= \hat{\dot{z}}_1^{(1)} + 0.5\, \Delta t\, \ddot{z}_1^{(1)} = 0.78737 + 0.5 \cdot 10^{-4} \cdot (-2221.6516) = 0.6763 \\
z_2^{(1)} &= \hat{z}_2^{(1)} + 0.25\, \Delta t^2\, \ddot{z}_2^{(1)} = 0.56808 \cdot 10^{-4} + 0.25 \cdot (10^{-4})^2 \cdot (-12874.1985) = 0.49245 \cdot 10^{-4} \\
\dot{z}_2^{(1)} &= \hat{\dot{z}}_2^{(1)} + 0.5\, \Delta t\, \ddot{z}_2^{(1)} = 1.1362 + 0.5 \cdot 10^{-4} \cdot (-12874.1985) = 0.4925
\end{aligned}$$