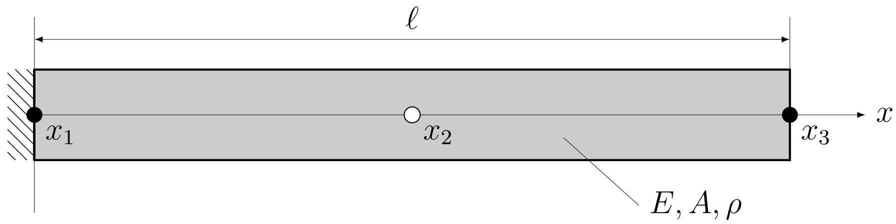
Problem set 10

Problem 1

We consider the dynamic response of a one-dimensional elastic bar of length ℓ, which is clamped at its left end and free at the right. The bar possesses a constant cross-sectional area A, Young's modulus E, and mass density ρ. It is assumed to be initially at rest, with zero displacement and velocity at time $t = 0$.
The bar is discretized using a single quadratic finite element with three equally spaced nodes. The configuration, including external dampers, is shown in figure below.



Assuming a diagonal lumped mass matrix obtained by summing the columns of the consistent mass matrix, the structure stiffness and lumped mass matrices are given by:

$$\mathbf{K} = \frac{EA}{3\ell}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \qquad \text{and} \qquad \mathbf{M} = \frac{\rho A \ell}{6}\text{diag}(1, 4, 1).$$

1. Identify the unconstrained and constrained degrees of freedom of the structure.
2. Assume an impulse force
$$\mathbf{r}(t) = \begin{bmatrix} 0 \\ 0 \\ \delta(t) \end{bmatrix}$$
is applied at the free end. Here $\delta(t)$ denotes the Dirac delta function. Using modal superposition, write the exact expression for the time response $\mathbf{q}(t)$ of the system.
Hint: the mode shapes, normalized with respect to the diagonal mass matrix, are:
$$\mathbf{p}_1 = \frac{1}{\sqrt{\rho A \ell}}\begin{bmatrix} 1.0066 \\ 1.3952 \end{bmatrix}, \quad \mathbf{p}_2 = \frac{1}{\sqrt{\rho A \ell}}\begin{bmatrix} -0.6976 \\ 2.0133 \end{bmatrix}$$
3. Consider the following data for the geometric and material properties of the bar:

$\ell = 1\,\text{m}$	(length)	$E = 2.1 \times 10^{11}\,\text{Pa}$	(Young's modulus)
$A = 0.0001\,\text{m}^2$	(cross-sectional area)	$\rho = 7850\,\text{kg/m}^3$	(mass density)

Using the average acceleration Newmark method ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$), compute the approximate solution to the modal equations at the initial time t = 0 and at the first time step $\Delta t = 0.1$ ms. This includes evaluating: the modal displacements $z_1(t)$, $z_2(t)$, the modal velocities $\dot{z}_1(t)$, $\dot{z}_2(t)$, and the modal accelerations $\ddot{z}_1(t)$, $\ddot{z}_2(t)$, at $t = 0$ and $t = \Delta t$, based on the initial conditions and the discretized Newmark update scheme. Use the following approximation for the delta function

$$\delta(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t > 0. \end{cases}$$

Note that this is a numerical approximation of the Dirac delta function used solely to compute the initial response within the time-stepping scheme. It induces an instantaneous velocity jump at t = 0, and in general, such approximations should be avoided in favor of more physically realistic or regularized loading models.