Linear elastodynamics

Strong and weak forms

ME473 Dynamic finite element analysis of structures

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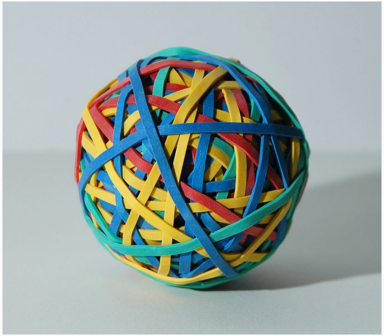
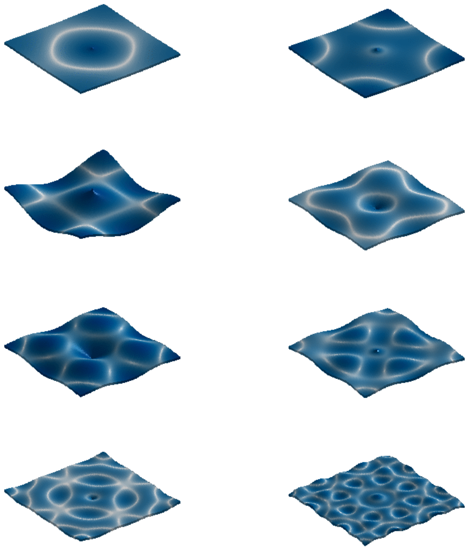
2025





(Credit: Noé Jiménez)

Linear: infinitesimally small deformations relative to the solid's size.

Elasticity: deformable material that returns to its original shape and size when the forces causing the deformation are removed.

Dynamics: study bodies in motion under the influence of the mechanical actions applied to them (time-dependent problems).

Summary

- Underlying hypothesis and notations
- Equilibrium equations of motion
- Strong formulation of equilibrium equations
- Weak formulation of equilibrium equations
- Example 1: longitudinal vibrations of bars
- Example 2: transversal vibrations of beams

Recommended readings

1. Gmür, Dynamique des structures (§2.1 and §2.2) [GM]
2. Gmür, Méthode des éléments finis (§ 5.1 and § 5.2) [GM1]
3. Neto et al., Engineering Computation of Structures (§ 1.1, §1.3 and §2.1) [N]

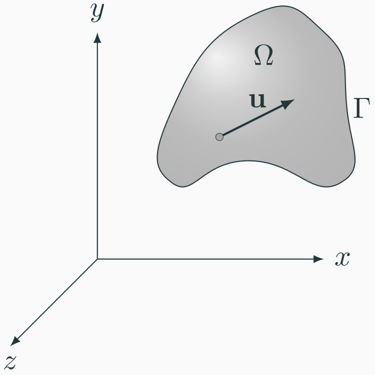
Underlying hypothesis and notations

Hypothesis and notations

- Inertial orthonormal reference frame $O(x, y, z)$. Sometimes will use $O(x_1, x_2, x_3)$.
- Continuous three-dimensional deformable finite (compact) **body** with volume Ω and surface Γ.
- **Material**: continuous and homogenous.
- **Deformations**: small (proportional to stress).
- Continuous and derivable time-dependent (*unknown*) **displacement field**:

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} u_1(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) \\ u_3(\mathbf{x}, t) \end{pmatrix}$$

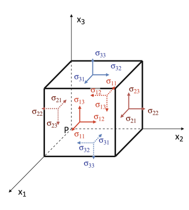
where $\mathbf{x} \in \bar{\Omega}$ and $t \in [0, T]$.



Stress and strain tensors

Stress tensor

$$\boldsymbol{\sigma}(\mathbf{x},t) = \begin{pmatrix} \sigma_{11}(\mathbf{x},t) \\ \sigma_{22}(\mathbf{x},t) \\ \sigma_{33}(\mathbf{x},t) \\ \sigma_{23}(\mathbf{x},t) \\ \sigma_{31}(\mathbf{x},t) \\ \sigma_{12}(\mathbf{x},t) \end{pmatrix}$$



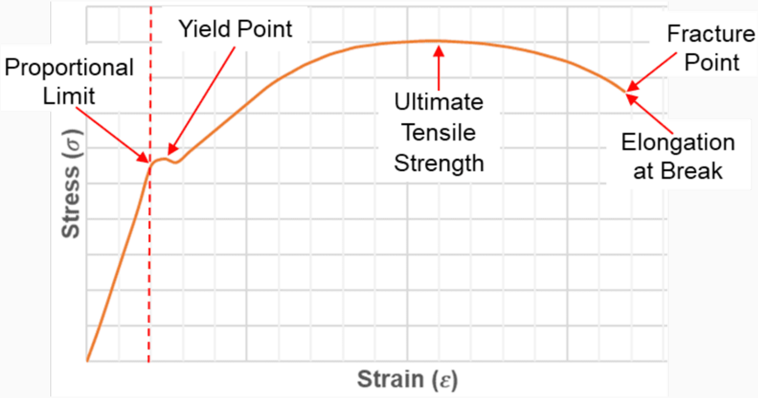
Credit: [N]

Strain tensor

$$\boldsymbol{\varepsilon}(\mathbf{x},t) = \begin{pmatrix} \varepsilon_{11}(\mathbf{x},t) \\ \varepsilon_{22}(\mathbf{x},t) \\ \varepsilon_{33}(\mathbf{x},t) \\ 2\varepsilon_{23}(\mathbf{x},t) \\ 2\varepsilon_{31}(\mathbf{x},t) \\ 2\varepsilon_{12}(\mathbf{x},t) \end{pmatrix}$$

- Stress expresses the internal forces that neighboring particles of a material exert on each other (SI units of Pascal.)
- Strain is defined as relative deformation, compared to a reference position configuration (SI units of meter per meter.)
- First index specifies the normal of the surface on which the stress/strain is acting, the second index specifies the direction of the stress/strain.
- if $i \neq j$ then $\sigma_{ij} = \tau_{ij}$ (shear stress) and $2\varepsilon_{ij} = \gamma_{ij}$ (engineering shear strain).
- Symmetric tensors: $\tau_{ij} = \tau_{ji}$ (Cauchy's stress theorem) and $\gamma_{ij} = \gamma_{ji}$.

The stress-strain curve



Credit: Fidelis

Generalised Hooke's law

The constitutive stress-strain relationship of linear elasticity is given by the Hooke's law:

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{C}\,\boldsymbol{\varepsilon}(\mathbf{x}, t) \quad \text{and} \quad \boldsymbol{\varepsilon}(\mathbf{x}, t) = \mathbf{S}\,\boldsymbol{\sigma}(\mathbf{x}, t)$$

where $\mathbf{C}$ is called the stiffness material matrix and $\mathbf{S} = \mathbf{C}^{-1}$.

1 - Anisotropic materials

No symmetry: these materials have properties that vary with direction. Their strength, stiffness, and conductivity differ depending on the axis of measurement.

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix}$$

- Most of the elements are non-zero, the matrix is considered dense.
- 21 independent constants (symmetry).

2 - Orthotropic materials

3 planes of symmetry: a special case of anisotropic materials where properties vary along three mutually perpendicular directions. These materials have three different principal material properties along three axes.

$$\mathbf{S} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix}$$

- E_i is the Young's modulus along axis Ox_i.
- G_{ij} is the shear modulus in direction Ox_i on the plane whose normal is Ox_j.
- ν_{ij} is the Poisson's ratio that corresponds to a contraction in direction Ox_j when an extension is applied in direction Ox_i.

3 - Isotropic materials

∞ **planes of symmetry**: these materials have identical properties in all directions. Their mechanical and physical properties do not change regardless of the direction in which they are measured.

$$\mathbf{S} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix}$$

- E is the Young's modulus, G is the shear modulus, and ν is the Poisson's ratio.
- Only two independent constants since $E = 2G(1+\nu)$.

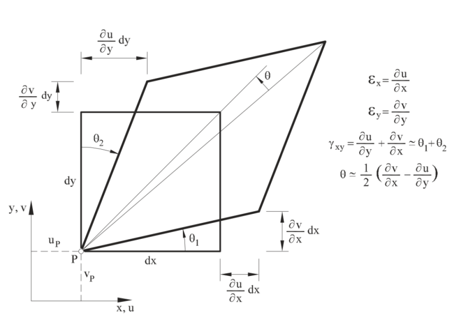
Strain-displacement relationship

- Both normal strain and shear strain can be regarded as a rate of displacement variation and angle per unit length.
- The components of strain can be obtained by derivatives of the displacements for small deformation in solids.
- The strain-displacement relation can be written in the three equivalent forms as follows:

a) $\boldsymbol{\varepsilon}(\mathbf{x},t) = \nabla \mathbf{u}(\mathbf{x},t)$

b)
$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ 0 & \partial_z & \partial_y \\ \partial_z & 0 & \partial_x \\ \partial_y & \partial_x & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

c) $\varepsilon_{ii} = \partial_{x_i} u_i$ and $2\varepsilon_{ij} = \partial_{x_i} u_j + \partial_{x_j} u_i$



Credit: [N]

Equilibrium equations of motion

Differential equations of movement

- Consider an infinitely small cube subjected to a force represented by the vector:

$$\mathbf{f}(\mathbf{x},t) = \begin{pmatrix} f_1(\mathbf{x},t) \\ f_2(\mathbf{x},t) \\ f_3(\mathbf{x},t) \end{pmatrix}.$$

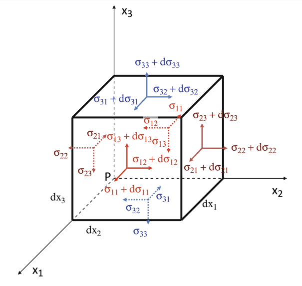
- Stress is not uniform. The variation of the stress between two opposite sides is linear and

$$d\sigma_{ij} = \partial_{x_k}\sigma_{ij}\, dx_k$$

- Newton 2nd law of motion taking into account internal forces $\mathbf{f}^i$ is

$$\mathbf{f}^i(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t) = \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t)\, dxdydz$$

where $\rho(\mathbf{x})$ is the material density, $\ddot{\mathbf{u}} = \partial_{tt}u$ is the acceleration, $dxdydz$ is the volume of the cube.



Credit: [N]

Differential equations of movement

- The equilibrium equation of elastodynamics in matrix form is:

$$\nabla^T \boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) = \rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t)$$

- Using the strain-displacement relation we can write it in terms of the displacement filed:

$$\nabla^T \mathbf{C} \nabla \mathbf{u}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) = \rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t)$$

Boundary conditions

Boundary conditions are appointed for each point on the solid surface $\Gamma = \Gamma_u \cup \Gamma_\sigma$.

- **Prescribed displacement:**

$$\mathbf{u} = \hat{\mathbf{u}} \qquad \text{sur } \Gamma_u \times]0, T[$$

where $\hat{\mathbf{u}} = \{\hat{u}_1, \hat{u}_2, \hat{u}_3\}^T$ is the given displacement prescribed on Γ_u.

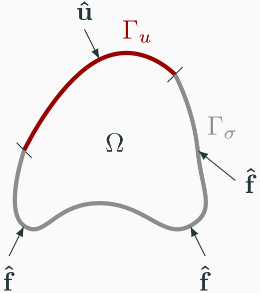
- **Prescribed surface force:**

$$\mathbf{N}^T \boldsymbol{\sigma} = \hat{\mathbf{f}} \qquad \text{sur } \Gamma_\sigma \times]0, T[$$

where $\hat{\mathbf{f}} = \{\hat{f}_1, \hat{f}_2, \hat{f}_3\}^T$ is the given surface load prescribed on Γ_σ and

$$\mathbf{N}^T = \begin{bmatrix} n_1 & 0 & 0 & 0 & n_3 & n_2 \\ 0 & n_2 & 0 & n_3 & 0 & n_1 \\ 0 & 0 & n_3 & n_2 & n_1 & 0 \end{bmatrix}$$

n_1, n_2 and n_3 being the direction cosines for the outward-pointing normal $\mathbf{n}$ to the boundary Γ_σ.



Initial conditions

The initial conditions are set at $t = 0$:

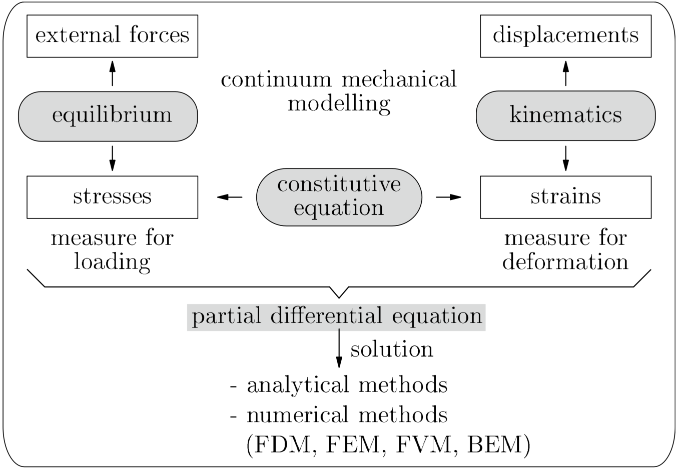
- **Imposed initial displacement field:**

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u_0}(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega$$

- **Imposed initial velocity field:**

$$\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v_0}(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega$$

Continuum mechanical modelling



Credit: A. Öchsner - PDE for classical structural members

Strong formulation of equilibrium equations

Strong form of elastodynamics

Given a deformable solid Ω with boundary Γ, and

- stiffness material matrix $\mathbf{C}$ and material density ρ,
- vector of body load $\mathbf{f}$ applied on Ω,
- prescribed boundary displacement $\hat{\mathbf{u}}$ on Γ_u and surface load $\hat{\mathbf{f}}$ on Γ_σ,
- prescribed initial (at $t = 0$) displacement $\mathbf{u}_0$ and initial velocity $\mathbf{v}_0$.

find the displacement $\mathbf{u} \in C^2(\bar{\Omega} \times [0, T], \mathbb{R}^3)$ such that

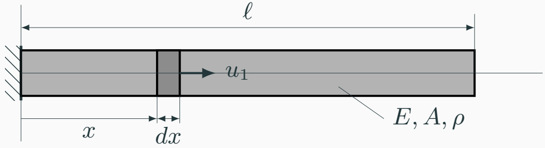
$$\begin{cases} \nabla^T \mathbf{C} \nabla \mathbf{u}(x,t) + \mathbf{f}(x,t) = \rho(x)\ddot{\mathbf{u}}(x,t) & \forall (\mathbf{x}, t) \in \Omega \times]0, T[\\ \mathbf{u}(x,t) = \hat{\mathbf{u}}(x,t) & \forall (\mathbf{x}, t) \in \Gamma_u \times]0, T[\\ \mathbf{N}^T \mathbf{C} \nabla \mathbf{u}(x,t) = \hat{\mathbf{f}}(x,t) & \forall (\mathbf{x}, t) \in \Gamma_\sigma \times]0, T[\\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & \forall \mathbf{x} \in \Omega \\ \dot{\mathbf{u}}(x,0) = \mathbf{v}_0(x) & \forall \mathbf{x} \in \Omega \end{cases}$$

Example 1: longitudinal vibrations of bars

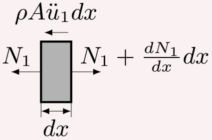
Strong form for longitudinal vibrations of bars

Kinematic assumptions:

- The bar cross-section is infinitely rigid in its own plane, remaining plane after deformation.
- Loads, which are uniform in every cross-section, can only be applied axially.
- The influence on the axial movement of the bar of lateral displacements due to the Poisson effect is negligible ($\sigma_{22} = \sigma_{33} = 0$). Thus the bar can undergo only axial stress σ_{11}, which is uniform in every cross-section.



- A cross-sectional area
- E Young's modulus
- ρ material density
- ℓ length
- x axial coordinate
- $u_1(x,t)$ axial displacement
- $N_1(x,t)$ normal stress



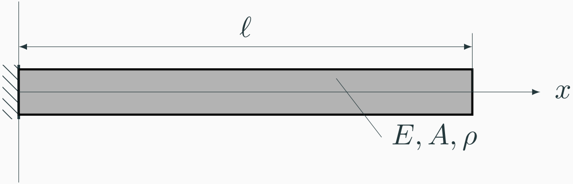
Equilibirum equation:

$$N_1 + \partial_x N_1 dx - N_1 = \rho A \ddot{u}_1 dx$$

Stress-strain-displacement relation:

$$N_1 = A\sigma_{11} = EA\varepsilon_{11} = EA\partial_x u_1$$

Strong form for longitudinal vibrations of bars



- A cross-sectional area
- E Young's modulus
- ρ material density
- ℓ length
- x axial coordinate
- $u_1(x,t)$ axial displacement

Find $u_1 \in C^2([0,l] \times [0,T])$ such that

$$EA\partial^2_{xx}u_1(x,t) = \rho A\ddot{u}_1(x,t) \quad \forall (x,t) \in]0,\ell[\times]0,T[$$

boundary conditions:

$$u_1(0,t) = 0 \qquad \forall t \in]0,T[$$
$$EA\partial_x u_1(\ell,t) = 0 \qquad \forall t \in]0,T[$$

initial conditions:

$$u_1(x,0) = u_0(x) \qquad \forall x \in]0,\ell[$$
$$\dot{u}_1(x,0) = v_0(x) \qquad \forall x \in]0,\ell[$$

Disclaimer - exact or closed-form solution

The strong form for longitudinal vibrations of a bar admits an exact solution in the following decoupled form

$$u_1(x,t) = \sum_{k=1}^{\infty} v_k(x)\phi_k(t)$$

where

$$v_k(x) = \sin\left(\frac{\pi(2k+1)}{2\ell}x\right)$$
$$\phi_k(t) = a_k \sin(\omega_k t) + b_k \cos(\omega_k t)$$

$$\omega_k = \frac{\pi(2k+1)}{2\ell}\sqrt{\frac{E}{\rho}}$$

- a_k and b_k depends respectively on the initial conditions u_0 and v_0.
- k represents a mode of vibration ($k = 1$ first or *fundamental* mode).

Weak formulation of equilibrium equations

Road to weak formulation

- Strong formulations lead to strong solutions in the sense that they require strong continuity in the field variables.
- The weak form is often expressed as an integral equation that requires weaker continuity on the variables.
- The weak form of the elastodynamics problem can be obtained using the *Virtual Work Principle*.

Through the following steps, we can obtain a weak form for a set of differential equations:

1. Multiply each differential equation by an appropriate arbitrary function
2. Integrate over the space domain of the problem.
3. Reduce the order of the involved derivatives using the divergence theorem.
4. Apply the boundary conditions reasonably.

Derivation of weak formulation

1. Introduce an admissible (avoid divergence of integral) **virtual displacement**:

$$\boldsymbol{\delta}\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \delta u_1(\mathbf{x}) \\ \delta u_2(\mathbf{x}) \\ \delta u_3(\mathbf{x}) \end{pmatrix}$$

2. Multiply the differential equation by $\boldsymbol{\delta}\mathbf{u}^T$ and integrate it over the spatial domain Ω.

$$\int_\Omega \boldsymbol{\delta}\mathbf{u}^T (\nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f})\, d\Omega = \int_\Omega \rho \boldsymbol{\delta}\mathbf{u}^T \ddot{\mathbf{u}}\, d\Omega$$

3. Apply the divergence theorem to the first term:

$$-\int_\Omega (\nabla \boldsymbol{\delta}\mathbf{u})^T \mathbf{C} \nabla \mathbf{u}\, d\Omega + \int_\Gamma \boldsymbol{\delta}\mathbf{u}^T \mathbf{N}^T \mathbf{C} \nabla \mathbf{u}\, d\Gamma + \int_\Omega \boldsymbol{\delta}\mathbf{u}^T \mathbf{f}\, d\Omega = \int_\Omega \rho \boldsymbol{\delta}\mathbf{u}^T \ddot{\mathbf{u}}\, d\Omega$$

4. Use the boundary conditions: $\mathbf{N}^T \mathbf{C} \nabla \mathbf{u} = \hat{\mathbf{f}}$ on Γ_σ and impose $\boldsymbol{\delta}\mathbf{u} = 0$ on Γ_u

$$-\int_\Omega (\nabla \boldsymbol{\delta}\mathbf{u})^T \mathbf{C} \nabla \mathbf{u}\, d\Omega + \int_{\Gamma_\sigma} \boldsymbol{\delta}\mathbf{u}^T \hat{\mathbf{f}}\, d\Gamma + \int_\Omega \boldsymbol{\delta}\mathbf{u}^T \mathbf{f}\, d\Omega = \int_\Omega \rho \boldsymbol{\delta}\mathbf{u}^T \ddot{\mathbf{u}}\, d\Omega$$

Functional spaces

To ensure the integrals remain finite, we impose that:

$$\mathbf{u} \in \mathcal{U} \qquad \text{and} \qquad \boldsymbol{\delta}\mathbf{u} \in \mathcal{V}$$

$$\mathcal{U} = \left\{ \mathbf{u}(\cdot, t) \in H^1(\Omega, \mathbb{R}^3) \mid \mathbf{u}(\cdot, t) = \hat{\mathbf{u}} \text{ on } \Gamma_u \; \forall t \in]0, T[\right\}$$

$$\mathcal{V} = \left\{ \mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \mid \mathbf{v} = 0 \text{ on } \Gamma_u \right\}$$

- Recall that $H^1(\Omega)$ is the Sobolev space defined as

$$H^1(\Omega) = \left\{ \mathbf{w} \in L^2(\Omega, \mathbb{R}^3) \mid \int_\Omega (\nabla \mathbf{w})^T \nabla \mathbf{w} \; d\Omega < \infty \right\}.$$

- Note that the difference between spaces $\mathcal{U}$ and $\mathcal{V}$ is that only the functions in $\mathcal{U}$ are time-dependent.
- The spaces are designed to incorporate only the displacement boundary condition on Γ_u.

Weak form of elastodynamics

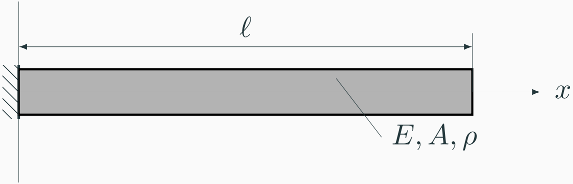
Given Ω, Γ, $\mathbf{C}$, ρ, $\mathbf{f}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{f}}$, $\mathbf{u}_0$, $\mathbf{v}_0$ as in the previous slide, find the displacement $\mathbf{u} \in \mathcal{U}$ such that that for any virtual displacement $\boldsymbol{\delta}\mathbf{u} \in \mathcal{V}$ we have

$$\int_\Omega (\nabla \boldsymbol{\delta}\mathbf{u})^T \mathbf{C} \nabla \mathbf{u} \, d\Omega + \int_\Omega \rho \, \boldsymbol{\delta}\mathbf{u}^T \ddot{\mathbf{u}} \, d\Omega = \int_{\Gamma_\sigma} \boldsymbol{\delta}\mathbf{u}^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega \boldsymbol{\delta}\mathbf{u}^T \mathbf{f} \, d\Omega,$$

$$\left. \begin{aligned} \int_\Omega \rho \, \boldsymbol{\delta}\mathbf{u}^T \mathbf{u}\big|_{t=0} \, d\Omega &= \int_\Omega \rho (\boldsymbol{\delta}\mathbf{u})^T \mathbf{u}_0 \, d\Omega, \\ \int_\Omega \rho \, \boldsymbol{\delta}\mathbf{u}^T \dot{\mathbf{u}}\big|_{t=0} \, d\Omega &= \int_\Omega \rho (\boldsymbol{\delta}\mathbf{u})^T \mathbf{v}_0 \, d\Omega. \end{aligned} \right\} \text{Initial conditions}$$

Example 1: longitudinal vibrations of bars

Strong form for longitudinal vibrations of bars (reminder)



- A cross-sectional area
- E Young's modulus (isotropic bar)
- ρ material density
- ℓ length
- x axial coordinate
- $u_1(x,t)$ axial displacement

Find $u_1 \in C^2([0,\ell] \times [0,T])$ such that

$$EA\partial_{xx}^2 u_1(x,t) = \rho A \ddot{u}_1(x,t) \quad \forall (x,t) \in]0,\ell[\times]0,T[$$

boundary conditions:

$$u_1(0,t) = 0 \qquad \forall t \in]0,T[$$
$$EA\partial_x u_1(\ell,t) = 0 \qquad \forall t \in]0,T[$$

initial conditions:

$$u_1(x,0) = u_0(x) \qquad \forall x \in]0,\ell[$$
$$\dot{u}_1(x,0) = v_0(x) \qquad \forall x \in]0,\ell[$$

Derivation of weak form for longitudinal vibrations of bars

1. Define the virtual displacement $\delta u_1(x)$ so that $\delta u_1 \in H^1(]0,\ell[)$ and $\delta u_1(0) = 0$.
2. Multiply the differential equation by the virtual displacement and integrate it over the intreval $]0,\ell[$.

$$\int_0^\ell EA\frac{\partial^2 u_1}{\partial x^2}\delta u_1 \; dx = \int_0^\ell \rho A \ddot{u}_1 \delta u_1 \; dx.$$

3. Use the integration by parts formula on the left hand side:

$$-\int_0^\ell EA\,\partial_x u_1\,\partial_x(\delta u_1)\; dx + \left[EA\partial_x u_1 \delta u_1\right]_0^\ell = \int_0^\ell \rho A \ddot{u}_1 \delta u_1 \; dx.$$

4. Make use of the boundary condition $EA\partial_x u_1(\ell,t) = 0$ to simplify

$$-\int_0^\ell EA\,\partial_x u_1\,\partial_x(\delta u_1)\; dx = \int_0^\ell \rho A \ddot{u}_1 \delta u_1 \; dx.$$

Weak form for longitudinal vibrations of bars

Find $u_1 \in \mathcal{U}$ such that $\forall \delta u_1 \in \mathcal{V}$ we have

$$\int_0^\ell EA\, \partial_x u_1\, \partial_x(\delta u_1)\ dx + \int_0^\ell \rho A \ddot{u}_1 \delta u_1\, dx = 0,$$

$$\left.\begin{aligned} \int_0^\ell \rho A u(x,0) \delta u_1(x)\, dx &= \int_0^\ell \rho A u_0(x) \delta u_1(x)\, dx, \\ \int_0^\ell \rho A \dot{u}(x,0) \delta u_1(x)\, dx &= \int_0^\ell \rho A v_0(x) \delta u_1(x)\, dx. \end{aligned}\right\} \text{Initial conditions}$$

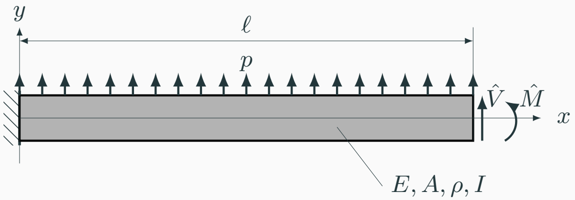
$$\mathcal{U} = \left\{ u_1(\cdot, t) \in H^1(]0,\ell[) \mid u_1(0,t) = 0\ \forall t \in]0,T[\right\}$$
$$\mathcal{V} = \left\{ \delta u_1 \in H^1(]0,\ell[) \mid \delta u_1(0) = 0 \right\}$$

Example 2: transversal vibrations of beams

Strong form for transversal vibrations of beams

Kinematic assumptions:

- The analysis will be restricted to the dynamic behavior of the beam in the $O(x, y)$ plane.
- A normal cross-section to the neutral fiber remains planar after deformation, but not necessarily orthogonal to it
- Shear deformations ε_{12} of sections are taken into account (Timoshenko or thick beam).



Model parameters:

- A cross-sectional area
- E Young's modulus
- ρ material density
- I moment of inertia
- ℓ length

Loads:

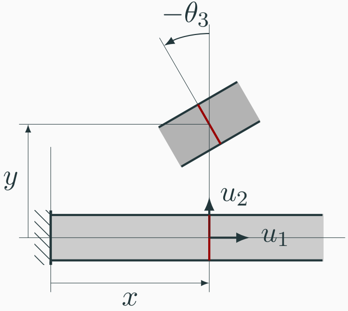
- $\hat{M}$ bending moment at free end
- $\hat{V}$ shear force at free end
- p distributed transversal load

Variables:

- $u_1(x, t)$ axial displacement
- $u_2(x, t)$ transversal displacement

Strong form for transversal vibrations of beams

Introduce an auxiliary variable $\theta_3(x,t)$ representing the total rotation of the section around the Oz axis.

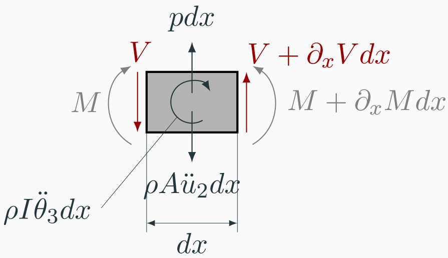


$$u_1 = -y\theta_3$$

Strain-displacement relationships

$$\varepsilon_{11} = \partial_x u_1 = -y\partial_x\theta_3$$
$$\varepsilon_{12} = \partial_x u_2 + \partial_y u_1 = \partial_x u_2 - \theta_3$$

Equilibirum equations of motion



$$\partial_x V + p = \rho A\ddot{u}_2$$
$$\partial_x M + V = \rho I\ddot{\theta}_3$$

Since $\sigma_{11} = E\varepsilon_{11}$ and $\sigma_{12} = kG\varepsilon_{12}$, where the constant k is the shear correction factor (parabolic distribution of tangential stresses)

$$V = \int_A \sigma_{12}\,dA = kGA\varepsilon_{12} = kGA(\partial_x u_2 - \theta_3)$$
$$M = -\int_A y\sigma_{11}\,dA = -\int_A yE\varepsilon_{11}\,dA = EI\partial_x\theta_3$$

Strong form for transversal vibrations of beams

The strong form for transversal vibrations of beams consists of finding the functions $u_2 \in C^2([0,\ell] \times [0,T])$ and $\theta_3 \in C^2([0,\ell] \times [0,T])$ such that the following equilibirum equations, boundary and initial conditions are satisfied.

Equilibirum equations

$$\partial_x\Big(kGA(\partial_x u_2 - \theta_3)\Big) + p = \rho A \ddot{u}_2$$

$$\partial_x\Big(EI\partial_x\theta_3\Big) + kGA(\partial_x u_2 - \theta_3) = \rho I \ddot{\theta}_3$$

In matrix form:

$$\underbrace{\begin{pmatrix} \partial_x & 0 \\ 1 & \partial_x \end{pmatrix}}_{\nabla_\sigma^T} \underbrace{\begin{pmatrix} kGA & 0 \\ 0 & EI \end{pmatrix}}_{\mathbf{C}} \underbrace{\begin{pmatrix} \partial_x & -1 \\ 0 & \partial_x \end{pmatrix}}_{\nabla_u} \underbrace{\begin{pmatrix} u_2 \\ \theta_3 \end{pmatrix}}_{\mathbf{u}} + \underbrace{\begin{pmatrix} p \\ 0 \end{pmatrix}}_{\mathbf{f}} = \underbrace{\begin{pmatrix} \rho A & 0 \\ 0 & \rho I \end{pmatrix}}_{\mathbf{M}} \underbrace{\begin{pmatrix} \ddot{u}_2 \\ \ddot{\theta}_3 \end{pmatrix}}_{\ddot{\mathbf{u}}}$$

$$\nabla_\sigma^T \mathbf{C} \nabla_u \mathbf{u} + \mathbf{f} = \mathbf{M}\ddot{\mathbf{u}}$$

Boundary and initial conditions

Boundary conditions

$$u_2(0,t) = 0 \qquad \forall t \in]0,T[\qquad\qquad kGA\big(\partial_x u_2(\ell,t) - \theta_3(\ell,t)\big) = \hat{V} \qquad \forall t \in]0,T[$$

$$\theta_3(0,t) = 0 \qquad \forall t \in]0,T[\qquad\qquad EI\partial_x \theta_3(\ell,t) = \hat{M} \qquad \forall t \in]0,T[$$

In matrix form:

$$\mathbf{u}(0,t) = 0 \qquad \forall t \in]0,T[$$

$$\mathbf{C}\nabla_u \mathbf{u}(\ell,t) = \hat{\mathbf{f}} \qquad \forall t \in]0,T[$$

Initial conditions

$$u_2(x,0) = u_0(x) \qquad \forall x \in]0,\ell[\qquad\qquad \dot{u}_2(x,0) = v_0(x) \qquad \forall x \in]0,\ell[$$

$$\theta_3(x,0) = \theta_0(x) \qquad \forall x \in]0,\ell[\qquad\qquad \dot{\theta}_3(x,0) = \phi_0(x) \qquad \forall x \in]0,\ell[$$

In matrix form:

$$\mathbf{u}(x,0) = \mathbf{u_0} \qquad \forall x \in]0,\ell[$$

$$\dot{\mathbf{u}}(x,0) = \mathbf{v_0} \qquad \forall t \in]0,\ell[$$