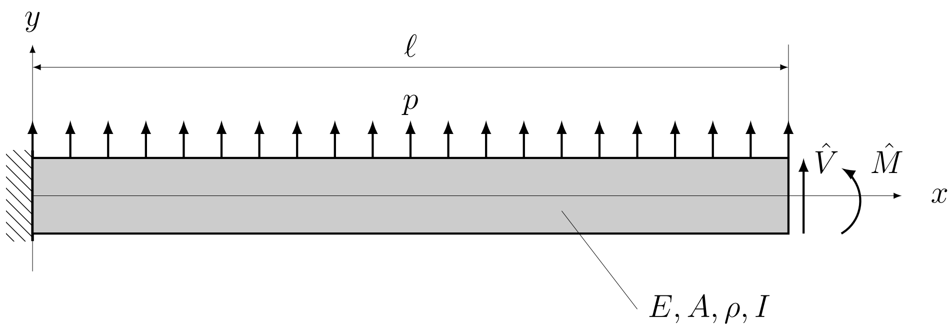
Problem set 1

Problem 1

Consider a rectilinear beam fixed at one end, subjected to a distributed force p and experiencing a bending moment $\hat{M}$ and a shear force $\hat{V}$ at the free end. The structure is characterised by a length l, a uniform cross-section A of moment of inertia I, a modulus of elasticity E and a mass density ρ.



Given the strong form of the governing equations of motion, determine the weak form describing the transversal vibrations of the beam governed by the Timoshenko beam theory, which accounts for both shear deformation and rotary inertia.
The equations of motion for the beam in Timoshenko theory are given by:

$$\begin{cases} \nabla_\sigma^T \mathbf{C} \nabla_u \mathbf{u} + \mathbf{f} = \mathbf{M}\ddot{\mathbf{u}} & \forall (x,t) \in]0,\ell[\times]0,T[\\ \mathbf{u}(0,t) = 0 & \forall t \in]0,T[\\ \mathbf{C}\nabla_u \mathbf{u}(l,t) = \hat{\mathbf{f}} & \forall t \in]0,T[\\ \mathbf{u}(x,0) = \mathbf{u}_0 & \forall x \in]0,\ell[\\ \dot{\mathbf{u}}(x,0) = \mathbf{v_0} & \forall x \in]0,\ell[\end{cases}$$

where $\mathbf{u}(x,t) = \begin{pmatrix} u_2(x,t) \\ \theta_3(x,t) \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} p \\ 0 \end{pmatrix}$, $\hat{\mathbf{f}} = \begin{pmatrix} \hat{V} \\ \hat{M} \end{pmatrix}$, $\mathbf{u}_0(x) = \begin{pmatrix} u_0(x) \\ \theta_0(x) \end{pmatrix}$, $\mathbf{v}_0 = \begin{pmatrix} v_0(x) \\ \phi_0(x) \end{pmatrix}$, and

$$\nabla_\sigma = \begin{bmatrix} \partial_x & 1 \\ 0 & \partial_x \end{bmatrix}, \quad \nabla_u = \begin{bmatrix} \partial_x & -1 \\ 0 & \partial_x \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} kGA & 0 \\ 0 & EI \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix}.$$

Problem 2

Consider a rectangular membrane stretched between fixed supports, subjected to a distributed load p, and experiencing a uniform tension S. The strong form governing membrane's transverse vibrations is: find the scalar quantity $u_3(x,y,t) \in C^2(\Omega \times [0,T])$, which represents the transverse displacement of the membrane at any point (x,y) and time t, such that the following equation is verified

$$S\nabla^2 u_3 + p = \rho \ddot{u}_3 \quad \text{in } \Omega\times]0,T[$$

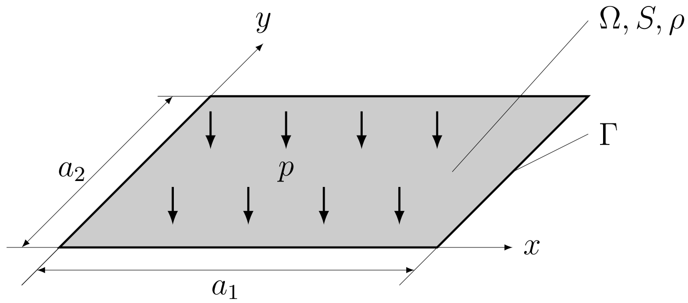
together with the following boundary and initial conditions:

$$\begin{cases} u_3(x,y,t) = 0 & (x,y) \in \Gamma,\ t \in]0,T[\\ u_3(x,y,0) = u_{03}(x,y) & (x,y) \in \Omega \\ \dot{u}_3(x,y,0) = v_{03}(x,y) & (x,y) \in \Omega. \end{cases}$$

Here the Laplacian operator is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

and ρ represents the density, $\Omega =]0, a_1[\times]0, a_2[$ is the domain (rectangle of sides a_1 and a_2) and Γ is the boundary of the membrane. The functions u_{03} and v_{03} specify the initial conditions of the membrane, where u_{03} represents the initial transverse displacement distribution, and v_{03} denotes the initial transverse velocity distribution.



Derive the corresponding weak form of the problem and define the appropriate functional spaces.

Problem 1 is taken from [G] examples 2..1.5 and 2.2.3
Problem 2 is taken from [G] exercise 2.6
[G] Gmür, Dynamique des structures: analyse modale numérique. EPFL Press, 1997