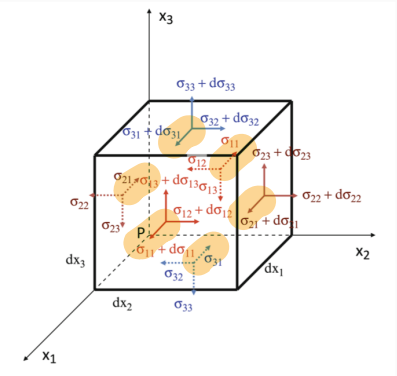
Differential equations of movement

- Consider an infinitely small cube subjected to a force represented by the vector:

$$\mathbf{f}(\mathbf{x},t) = \begin{pmatrix} f_1(\mathbf{x},t) \\ f_2(\mathbf{x},t) \\ f_3(\mathbf{x},t) \end{pmatrix}.$$



Credit: [N]

- Stress is not uniform. The variation of the stress between two opposite sides is linear and

$$d\sigma_{ij} = \partial_{x_k}\sigma_{ij}\, dx_k \qquad (*)$$

- Newton 2nd law of motion taking into account internal forces $\mathbf{f}^i$ is

$$\mathbf{f}^i(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t) = \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t)\, dxdydz$$

where $\rho(\mathbf{x})$ is the material density, $\ddot{\mathbf{u}} = \partial_{tt}u$ is the acceleration, $dxdydz$ is the volume of the cube.

Equilibrium of forces in the x_1 direction:

$$(\sigma_{11} + d\sigma_{11} - \sigma_{11})\, dx_2\, dx_3 + (\sigma_{21} + d\sigma_{21} - \sigma_{21})\, dx_1 dx_3 +$$

$$+ (\sigma_{31} + d\sigma_{31} - \sigma_{31})\, dx_1\, dx_2 + f_1\, dx_1 dx_2 dx_3 = \rho\, \ddot{u}_1 dx_1 dx_2 dx_3$$

After accounting $(*)$, the latter equation is:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = \rho\, \ddot{u}_1$$

Similarly, the equilibrium of forces in the x_2 and x_3 directions leads to:

$$\frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3} + f_j = \rho\, \ddot{u}_j \qquad (j = 1,2,3)$$

The equilibrium equations can be written in matrix form as

$$\nabla^T \sigma + f = \rho \ddot{u}$$

where $\nabla^T = \begin{pmatrix} \partial_x & 0 & 0 & 0 & \partial_z & \partial_y \\ 0 & \partial_y & 0 & \partial_z & 0 & \partial_x \\ 0 & 0 & \partial_z & \partial_y & \partial_x & 0 \end{pmatrix}$, $\sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}$.

Since $\sigma = C\varepsilon = C\nabla u$ we obtain

$$\boxed{\nabla^T C \nabla u + f = \rho \ddot{u}}$$

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1. Introduce an admissible (avoid divergence of integral) **virtual displacement**:

$$\boldsymbol{\delta u}(\mathbf{x}) = \begin{pmatrix} \delta u_1(\mathbf{x}) \\ \delta u_2(\mathbf{x}) \\ \delta u_3(\mathbf{x}) \end{pmatrix}$$

2. Multiply the differential equation by $\boldsymbol{\delta u}^T$ and integrate it over the spatial domain Ω.

$$\int_\Omega \boldsymbol{\delta u}^T (\nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f})\, d\Omega = \int_\Omega \rho \boldsymbol{\delta u}^T \ddot{\mathbf{u}}\, d\Omega$$

3. Apply the divergence theorem to the first term:

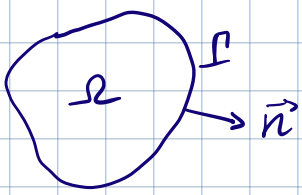
$$-\int_\Omega (\nabla \boldsymbol{\delta u})^T \mathbf{C} \nabla \mathbf{u}\, d\Omega + \int_\Gamma \boldsymbol{\delta u}^T \mathbf{N}^T \mathbf{C} \nabla \mathbf{u}\, d\Gamma + \int_\Omega \boldsymbol{\delta u}^T \mathbf{f}\, d\Omega = \int_\Omega \rho \boldsymbol{\delta u}^T \ddot{\mathbf{u}}\, d\Omega$$

Let's clarify how the term $\int_\Omega (\delta u)^T \nabla^T C \nabla u \, d\Omega$ is transformed into

$$-\int_\Omega (\nabla \delta u)^T C \nabla u \, d\Omega + \int_\Gamma (\delta u)^T N^T C \nabla u \, d\Gamma$$

by applying the divergence theorem:

Div. theorem: $\int_\Omega \partial_{x_i} u \, d\Omega = \int_\Gamma u \, n_i \, d\Gamma$



$\vec{n}$ outward pointing normal vector.

$$\int_\Omega (\delta u)^T \nabla^T C \nabla u \, d\Omega = \int_\Omega (\delta u)^T \nabla^T \sigma \, d\Omega$$

$$= \sum_{j=1}^{3} \int_\Omega \delta u_j \left(\frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3} \right) d\Omega$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega \delta u_j \frac{\partial \sigma_{ij}}{\partial x_i} \, d\Omega$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega \underbrace{\frac{\partial}{\partial x_i} (\delta u_j \sigma_{ij})}_{\text{apply div. thm.}} - \frac{\partial \delta u_j}{\partial x_i} \sigma_{ij} \, d\Omega$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Gamma \delta u_j \sigma_{ij} n_i \, d\Gamma - \int_\Omega \frac{\partial \delta u_j}{\partial x_i} \sigma_{ij} \, d\Omega$$

$$= \int_\Gamma (\delta u)^T N^T \sigma \, d\Gamma - \int_\Omega (\nabla \delta u)^T \sigma \, d\Omega$$

$$= \int_\Gamma (\delta u)^T N^T C \nabla u \, d\Gamma - \int_\Omega (\nabla \delta u)^T C \nabla u \, d\Omega$$