ME470: Problem sheet 4 solution: Membrane Mechanics

Alessandro Rizzi and Roxane Ollivier

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1 Cytoskeleton and cell membrane

a)

$$k_{sp} = \frac{3K_BT}{2l_pL_c} = 2 * 10^{-6}N/m \tag{1}$$

i)

$$\begin{aligned} K_A &= \frac{\sqrt{3}}{2}k_{sp} = \sqrt{3} * 10^{-6}N/m \\ \mu_A &= \frac{\sqrt{3}}{4}k_{sp} = \frac{\sqrt{3}}{2} * 10^{-6}N/m \end{aligned} \tag{2}$$

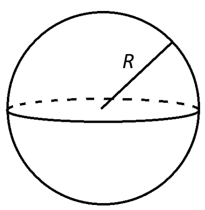
ii)

$$K_B = \frac{K_Ah^2}{12} = \frac{\sqrt{3}}{196} * 10^{-20}J \tag{3}$$

b)

$$K_B = \frac{10\sqrt{3}}{184} * (4 * 10^{-21})J \approx 0.022K_BT \tag{4}$$

c)



$$\begin{aligned} \Delta\pi &= \frac{2\gamma}{R} \\ K_A &= 2\gamma = R\Delta\pi \end{aligned}$$

$$\Delta\pi = \frac{\sqrt{3}}{6}Pa \approx 0.29Pa \tag{6}$$

2 Red blood cell shape energy

a)

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}$$

$$\frac{\partial \boldsymbol{x}}{\partial u} = (-(c + a\cos v)\sin u, (c + a\cos v)\cos u, 0)$$

$$\frac{\partial \boldsymbol{x}}{\partial v} = (-a\sin v\cos u, -a\sin v\sin u, 0)$$

Then, $E = (c + a\cos v)^2$, $F = 0$, $G = a^2$, $\Delta = a(c + a\cos v)$.
From second derivatives, $e = -\cos u(c + a\cos v)$, $f = 0$, $g = -a$.

$$H = \frac{-(c + 2a\cos v)}{2a(c + a\cos v)} \tag{7}$$

b)

$$U_{\text{bend}} = \frac{1}{2}k_b \int\int (2H)^2 dA + k_G \int\int K_G dA$$

Use Gauss-Bonnet theorem:

$$\int\int K_G dA = 4\pi$$

$$\int\int (2H)^2 dA = \int_0^{2\pi}\int_0^{2\pi} \frac{4(c + 2a\cos v)^2}{4a^2(c + a\cos v)^2} a(c + a\cos v)du dv = \frac{2\pi}{a}\int_0^{2\pi} \frac{(c + 2a\cos v)^2}{c + a\cos v} dv$$

Using the hint,

$$U_{\text{bend}} = \frac{2\pi^2 c^2 k_b}{a\sqrt{c^2 - a^2}} + 4\pi k_G = 285.57 k_B T. \tag{8}$$

c) For a sphere,

$$U_{\text{bend}} = 8\pi k_b + 4\pi k_G = 125.66 k_B T. \tag{9}$$

d) If no disk, then

$$\int\int K_G dA = 4\pi(1 - g) = 0, \tag{10}$$

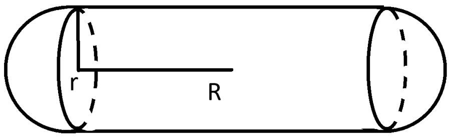
so the decrease in energy is $40\pi k_B T$.
e) Torus with $a = c$

$$\int\int K_G dA = 0,$$

$$\int\int (2H)^2 dA = \frac{4\pi^2 c^2}{a\sqrt{c^2 - a^2}} \to \infty \tag{11}$$

3 Energy of dividing cells

a)



$$r = 1\mu m$$
$$R = 2\mu m$$

$$U_{band} = \frac{1}{2} K_b \int\int (2H)^2 dA + K_G \int\int K_G dA \tag{13}$$

Using G-B theorem,

$$\int\int K_A dA = 4\pi \tag{14}$$

For spherical caps,

$$\begin{aligned} A_s &= 4\pi r^2 \\ H_s &= 1/r \end{aligned} \tag{15}$$

For cylinder,

$$\begin{aligned} A_c &= 4\pi R r \\ H_c &= 1/2r \end{aligned} \tag{16}$$

So,

$$U_{band} = \frac{1}{2} K_b \left(16\pi + \frac{4\pi R}{r}\right) + 4\pi K_G = 80 K_B T \tag{17}$$

b) Conservation of volume and r:

$$\frac{4}{3}\pi r^3 + \pi r^2 R = 2\left(\frac{4}{3}\pi r^3 + \pi r^2 R'\right) \tag{18}$$

$$R' = R/2 - r/3 = 2/3\mu m \tag{19}$$

$$\begin{aligned} U_{band} &= 2\left[\frac{1}{2} K_b \left(16\pi + \frac{4\pi R}{r}\right) + 4\pi K_G\right] \\ &= 2\left[\frac{1}{2} K_b \left(16\pi + \frac{4\pi R'}{r}\right) + 4\pi K_G\right] \\ &= 80 K_B T + \frac{60}{3}\pi K_B T \\ &= \frac{320}{3}\pi K_B T \end{aligned} \tag{20}$$

4 Sliding bilayers

For single bilayer

$$U_b = \frac{1}{2}k_b \int c^2 dA = \frac{1}{h}\int_{-h/2}^{h/2} \frac{1}{2}k_A \int \left(\frac{Z}{R}\right)^2 dA = \frac{k_A}{2h}\frac{h^3}{12}\int c^2 dA$$

Then, $k_b = \frac{1}{12}k_A h^2$.

If two monolayer, free sliding,

$$U_b = \frac{1}{2}k_b \int c^2 dA = 2\frac{1}{h}\int_{-h/4}^{h/4} \frac{1}{2}k_A \int \left(\frac{Z}{R}\right)^2 dA = \frac{2k_A}{2h}\frac{h^3}{96}\int c^2 dA$$

Then, $k_b = \frac{1}{48}k_A h^2$.