ME470: Problem sheet 3 - 2D network structure - correction

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1 A first look at rubber

Treat rubber as a 3d isotropic homogeneous linear elastic material.

a)

$$\nu = \frac{3K - 2\mu}{6K + 2\mu} = \frac{3K - \frac{1}{5000}\mu}{6K + \frac{1}{5000}\mu} = 0.49995$$

incompressible, no volume change

b)

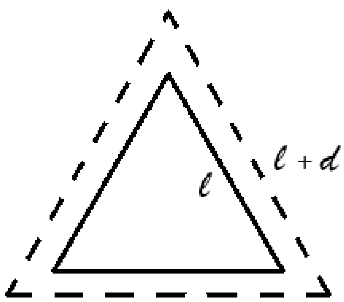
$$E = \frac{9K\mu}{3K + \mu} = \frac{\frac{9}{10000}K^2}{3K + \frac{K}{10000}} = \frac{9}{30001}K$$

easy to extend, but hard to change volume

2 Pre-stressed triangular network

a) Consider a triangular lattice where each bond between neighboring particles behaves like a spring with spring constant k_{sp} and rest length l_0.

For a uniform expansion, the variation of elastic energy stored in each bond is:



$$\Delta E_{elastic} = \frac{1}{2}k_{sp}\left((l + d - l_0)^2 - (l - l_0)^2\right)\frac{1}{2} \tag{1}$$

The work done by the pre-stress during a small extension ΔA of the area of the plaquette is:

$$W_{pre-stress} = \tau \Delta A = \tau\left(\frac{\sqrt{3}}{4}(l + d)^2 - \frac{\sqrt{3}}{4}l^2\right) \tag{2}$$

So the enthalpy variation is:

$$\begin{aligned}\Delta H &= \frac{3}{2} * \frac{1}{2} k_{sp} \left((l + d - l_0)^2 - (l - l_0)^2 \right) - \tau \left(\frac{\sqrt{3}}{4}(l+d)^2 - \frac{\sqrt{3}}{4} l^2 \right) \\ &= \frac{3}{4} k_{sp} \left[2d(l - l_0) + d^2 \right] - \tau \frac{\sqrt{3}}{4} \left[2ld - d^2 \right]\end{aligned} \tag{3}$$

$$\begin{aligned}\Delta h &= \frac{\Delta H}{(\frac{\sqrt{3}}{4} l^2)} \\ &= \sqrt{3} k_{sp} \left(2 \frac{d}{l} \frac{l - l_0}{l} + \left(\frac{d}{l} \right)^2 \right) - \tau \left(2 \frac{d}{l} + \left(\frac{d}{l} \right)^2 \right) \\ &= \sqrt{3} k_{sp} \left(2 \frac{d}{l} \frac{\tau}{\sqrt{3} k_{sp}} + \left(\frac{d}{l} \right)^2 \right) - \tau \left(2 \frac{d}{l} + \left(\frac{d}{l} \right)^2 \right)\end{aligned} \tag{4}$$

Because at equilibrium $\frac{l - l_0}{l} = \frac{\tau}{\sqrt{3} k_{sp}}$
So,

$$\Delta h = \sqrt{3} k_{sp} \left(1 - \frac{\tau}{\sqrt{3} ksp} \right) \left(\frac{d}{l} \right)^2 \tag{5}$$

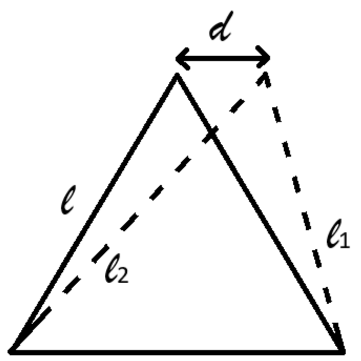
The strain energy density is:

$$\Delta h = 2 K_A \left(\frac{d}{l} \right)^2 \tag{6}$$

Therefore, with Eq.4 and Eq. 6, we get:

$$K_A = \frac{\sqrt{3}}{2} k_{sp} \left(1 - \frac{\tau}{\sqrt{3} k_{sp}} \right) \tag{7}$$

b) To compute the shear modulus μ_A for a pre-stressed triangular lattice, we'll follow a similar approach as we did for the bulk modulus K_A, but this time considering the system under simple shear deformation. We expand for $d << l$



$$\begin{aligned} l_1 &\approx l - \frac{d}{2} + \frac{3d^2}{8l} \\ l_2 &\approx l + \frac{d}{2} + \frac{3d^2}{8l} \end{aligned}$$

$$\begin{aligned}\Delta H &= \frac{1}{2} k_{sp} \left[\left(l - \frac{d}{2} + \frac{3d^2}{8l} - l_0 \right)^2 + \left(l + \frac{d}{2} + \frac{3d^2}{8l} - l_0 \right)^2 - 2(l - l_0)^2 \right] \frac{1}{2} \\ &\approx \frac{k_{sp}}{8} \left(1 + \frac{\sqrt{3}\tau}{k_{sp}} \right) d^2\end{aligned} \tag{9}$$

So

$$\Delta h = \frac{k_{sp}}{2\sqrt{3}} \left(1 + \frac{\sqrt{3}\tau}{k_{sp}}\right) \left(\frac{d}{l}\right)^2 \tag{10}$$

Moreover, the strain energy is:

$$\Delta h = 2\mu_A \epsilon_{xy}^2 = \frac{2}{3}\mu_A \left(\frac{d}{l}\right)^2 \tag{11}$$

Therefore, with Eq.10 and Eq. 11, we get:

$$\mu_A = \frac{\sqrt{3}}{4} k_{sp} \left(1 + \frac{3\tau}{k_{sp}}\right) \tag{12}$$

3 Poisson ratio of triangular lattice

a)

$$\begin{aligned}
\nu_A &= \frac{K_A - \mu_A}{K_A + \mu_A} \\
&= \frac{2\left(1 - \frac{\tau}{\sqrt{3}k_{sp}}\right) - \left(1 + \frac{\sqrt{3}\tau}{k_{sp}}\right)}{2\left(1 - \frac{\tau}{\sqrt{3}k_{sp}}\right) + \left(1 + \frac{\sqrt{3}\tau}{k_{sp}}\right)} \\
&= \frac{1 - \frac{5\sqrt{3}\tau}{3k_{sp}}}{3 + \frac{\sqrt{3}\tau}{3k_{sp}}}
\end{aligned}$$

b) $\nu_A < 0$ if $1 - \frac{5\sqrt{3}\tau}{3k_{sp}} < 0$, hence if $\tau > \frac{\sqrt{3}}{5} k_{sp}$.

c) at zero τ, $\mu_A = \frac{\sqrt{3}}{4} k_{sp}$. $\nu_A < 0$ if

$$\tau > \frac{\sqrt{3}}{5} k_{sp} = \frac{4}{5}\mu_A = 4 \cdot 10^{-6} \mathrm{J/m^2}$$