ME470: Problem sheet 1 - Continuum mechanics and statistical mechanics - correction

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1 Index notation

(a) For $\cos\theta$

$$\vec{a}\cdot\vec{a} = a_m a_m \quad \text{and} \quad \vec{b}\cdot\vec{b} = b_k b_k \tag{1}$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = a_i b_i \tag{2}$$

Thus,

$$\cos\theta = \frac{a_i b_i}{|\vec{a}||\vec{b}|} = \frac{a_i b_i}{\sqrt{a_m a_m}\sqrt{b_k b_k}} \tag{3}$$

For $|\sin\theta|$

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||\sin\theta| = \sqrt{\epsilon_{ijk}\epsilon_{fgk}a_i b_j a_f b_g} \tag{4}$$

Since,

$$\epsilon_{ijk}\epsilon_{fgk} = \delta_{if}\delta_{jg} - \delta_{ig}\delta_{jf} \tag{5}$$

We obtain,

$$\epsilon_{ijk}\epsilon_{fgk}a_i b_j a_f b_g = a_i a_i b_j b_j - a_i a_j b_i b_j \tag{6}$$

Thus,

$$|\sin\theta| = \frac{\sqrt{a_i a_i b_j b_j - a_i a_j b_i b_j}}{\sqrt{a_m a_m}\sqrt{b_n b_n}} \tag{7}$$

(b)

$$\begin{aligned} 2T_{ij}a_i a_j &= T_{ij}a_i a_j + T_{ij}a_i a_j \\ &= T_{ij}a_i a_j - T_{ji}a_i a_j \quad (\mathrm{T_{ij}} = -\mathrm{T_{ji}}) \\ &= T_{ij}a_i a_j - T_{ij}a_j a_i \\ &= 0 \end{aligned} \tag{8}$$

So,

$$T_{ij}a_ia_j = 0 \tag{9}$$

(c)

$$\begin{aligned} 2\epsilon_{ijk}a_ja_k &= \epsilon_{ijk}a_ja_k + \epsilon_{ijk}a_ja_k \\ &= \epsilon_{ijk}a_ja_k - \epsilon_{ikj}a_ja_k \quad (\epsilon_{\text{ijk}} = -\epsilon_{\text{ikj}}) \\ &= \epsilon_{ijk}a_ja_k - \epsilon_{ikj}a_ka_j \\ &= 0 \end{aligned} \tag{10}$$

So,

$$\epsilon_{ijk}a_ja_k = 0 \tag{11}$$

2 Homogeneous isotropic linear elastic material

(a) Considering a superposition of three uni-axial stresses we get

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz}) = \frac{1+\nu}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}),$$

and equivalent expressions for ϵ_{yy} and ϵ_{zz}. This can be generalized in the index notation to include non-diagonal elements as well

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{ll}\delta_{ij}. \tag{12}$$

(b) Using the fact that $\delta_{lm}\delta_{ml} = \delta_{ll} = \mathrm{Tr}(\mathbf{I}) = 3$ in a 3-dimensional space,

$$\epsilon_{ll} = \epsilon_{lm}\delta_{ml} = \frac{1+\nu}{E}\sigma_{lm}\delta_{ml} - \frac{\nu}{E}\sigma_{ii}\delta_{lm}\delta_{ml} \tag{13}$$

$$= \frac{1+\nu}{E}\sigma_{ll} - \frac{3\nu}{E}\sigma_{ll} = \frac{1-2\nu}{E}\sigma_{ll} \tag{14}$$

Then, inverting Eq. (12) yields

$$\sigma_{ij} = \frac{E}{1+\nu}\epsilon_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)}\epsilon_{ll}\delta_{ij} \tag{15}$$

(c) For simple shear, $\sigma_{xy} = 2\mu\epsilon_{xy}$. It immediately follows from Eq. (15) that

$$\mu = \frac{E}{2(1+\nu)}$$

Under isotropic stress, $\sigma_{ik} = -p\delta_{ik}$. Then $\sigma_{ll} = -3p$ and $\mathrm{d}\sigma_{ll} = -3\mathrm{d}p$.
By definition,

$$\frac{1}{K} = -\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}p}.$$

Then,

$$\epsilon_{ll} = \frac{\mathrm{d}V' - \mathrm{d}V}{\mathrm{d}V}, \quad \mathrm{d}\epsilon_{ll} = \frac{\mathrm{d}V}{V},$$

and

$$\frac{1}{K} = 3\frac{\mathrm{d}\epsilon_{ll}}{\mathrm{d}\sigma_{ll}} = \frac{3(1-2\nu)}{E},$$

where we have differentiated the final expression of Eq. (14). Then

$$K = \frac{E}{3(1-2\nu)}.$$

(d)

$$\frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{E}{3}\left(-\frac{1}{1+\nu} + \frac{1}{1-2\nu}\right) = -\frac{2}{3}\mu + K.$$

Using this and the expression of μ, Eq. (15) becomes

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \left(-\frac{2}{3}\mu + K\right)\epsilon_{ll}\delta_{ij} \tag{16}$$

$$= 2\mu\left(\epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\delta_{ij}\right) + K\epsilon_{ll}\delta_{ij}. \tag{17}$$

(e)

$$\sigma_{ll} = 3K\epsilon_{ll}, \quad \epsilon_{ll} = \frac{1}{3K}\sigma_{ll}$$

Inserting this in Eq. (17) gets us

$$\sigma_{ij} = 2\mu\left(\epsilon_{ij} - \frac{1}{9K}\sigma_{ll}\delta_{ij}\right) + \frac{1}{3}\sigma_{ll}\delta_{ij}.$$

Inverting this expression leads to the result

$$\epsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{1}{3}\sigma_{ll}\delta_{ij}\right) + \frac{1}{9K}\sigma_{ll}\delta_{ij}.$$

(f)

$$\begin{aligned} u &= \frac{1}{2}\epsilon_{ij}\sigma_{ij} \\ &= \frac{1}{2}\epsilon_{ij}\left(2\mu\left[\epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\delta_{ij}\right] + k\epsilon ll\delta_{ij}\right) \\ &= \mu\left[\epsilon_{ij}\epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\epsilon_{kk}\right] + \frac{K}{2}\epsilon_{ll}\epsilon_{kk}. \end{aligned} \tag{18}$$

The term in square brackets of the last line can be written as

$$\begin{aligned} &\left(\epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\delta_{ij}\right)\left(\epsilon_{ij} - \frac{1}{3}\epsilon_{kk}s_{ij}\right) \\ &= \epsilon_{ij}\epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\epsilon_{ii} - \frac{1}{3}\epsilon_{kk}\epsilon_{ii} + \frac{1}{9}\cdot\epsilon_{kk}\epsilon_{ll}\delta_{ii} \\ &= \epsilon_{ij}\epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\epsilon_{ll}, \end{aligned}$$

remembering that $\delta_{ii} = 3$. Then, inserting this result into Eq. (18) leads to

$$u = \mu\left(\epsilon_{ij} - \frac{1}{3}\epsilon_{ll}\delta_{ij}\right)^2 + \frac{K}{2}\epsilon_{ll}^2$$

3 2D elasticity

(a) First rewrite the energy as

$$\begin{aligned} u_A &= \mu_A \left(\epsilon_{ij}\epsilon_{ij} - \frac{1}{2}\epsilon_{ll}\epsilon_{kk} \right) + \frac{K_A}{2}\epsilon_{ll}\epsilon_{kk} \\ &= \mu_A \epsilon_{ij}\epsilon_{ij} + \frac{1}{2}\left(K_A - \mu_A\right)\epsilon_{ll}\epsilon_{kk}, \end{aligned}$$

keeping in mind that now $\delta_{ii} = 2$. Then

$$\begin{aligned} \tau_{ij} &= \frac{\partial u_A}{\partial \epsilon_{ij}} \\ &= \frac{\partial}{\partial \epsilon_{ij}} \left(\mu_A \epsilon_{ij}\epsilon_{ij} + \frac{1}{2}\left(K_A - \mu_A\right)\epsilon_{ll}\epsilon_{kk} \right) \\ &= 2\mu_A \epsilon_{ij} + \left(K_A - \mu_A\right)\epsilon_{ll}\delta_{ij} \end{aligned}$$

(b)

$$\begin{aligned} \tau_{ll} &= 2K_A \epsilon_{ll}, \\ \tau_{ij} &= 2\mu_A \epsilon_{ij} + \frac{K_A - \mu_A}{2K_A}\tau_{ll}\delta_{ij}. \end{aligned}$$

Inverting the last equation leads to the result

$$\epsilon_{ij} = \frac{1}{2\mu_A}\tau_{ij} + \frac{\mu_A - K_A}{4K_A\mu_A}\tau_{ll}\delta_{ij}.$$

(c) Assuming $\tau_{xx} = \tau$ and $\tau_{yy} = 0$, the strain diagonal components are

$$\begin{aligned} \epsilon_{xx} &= \frac{1}{2\mu_A}\tau + \frac{\mu_A - K_A}{4K_A\mu_A}\tau = \frac{K_A + \mu_A}{4K_A\mu_A}\tau, \\ \epsilon_{yy} &= \frac{\mu_A - K_A}{4K_A\mu_A}\tau. \end{aligned}$$

Then,

$$\nu_A = -\frac{\epsilon_{yy}}{\epsilon_{xx}} = \frac{K_A - \mu_A}{K_A + \mu_A}.$$

(d)

$$\begin{aligned} \tau_{xx} &= \frac{4K_A\mu_A}{K_A + \mu_A}\epsilon_{xx} = E_A \epsilon_{xx}. \\ E_A &= \frac{4K_A\mu_A}{K_A + \mu_A}. \end{aligned}$$

4 Two dice

(a) The number of microstates is equal to the total number of pairs of outcomes for each of the two dices $(i; j)$. So there are $6 * 6 = 36$ microstates. There are 11 macrostates, as the sum of the two dices numbers ranges from 2 (for the pair (1;1)) to 12 (for the pair (6;6)).
(b) The density of states for each macrostate are:

Macrostates	Density of states	Microstates
S=2	$\Omega(S = 2)$ = 1	(1;1)
S=3	$\Omega(S = 3)$ = 2	(1;2), (2;1)
S=4	$\Omega(S = 4)$ = 3	(1;3), (2;2), (3;1)
S=5	$\Omega(S = 5)$ = 4	.
S=6	$\Omega(S = 6)$ = 5	.
S=7	$\Omega(S = 7)$ = 6	.
S=8	$\Omega(S = 8)$ = 5	.
S=9	$\Omega(S = 9)$ = 4	.
S=10	$\Omega(S = 10)$ = 3	.
S=11	$\Omega(S = 11)$ = 2	.
S=12	$\Omega(S = 12)$ = 1	.

(c) The probability of each macrostate S is:

$$P(S) = \frac{\Omega(S)}{36} \tag{19}$$

5 Two state system

(a) The partition function is the sum of the Boltzmann factors for all possible states of the system:

$$\begin{aligned} Z &= \sum_{states} \exp\left(-\frac{E}{k_B T}\right) \\ &= \sum_{s=0}^{s=1} \exp\left(-\frac{E(s)}{k_B T}\right) \\ &= \exp\left(-\frac{\epsilon_{closed}}{k_B T}\right) + \exp\left(-\frac{(\epsilon_{open} - \tau \Delta A)}{k_B T}\right) \end{aligned} \tag{20}$$

where $E(s)$ is the energy of the system in state s.

For $s = 0$ (closed state),

$$E(0) = \epsilon_{closed} \tag{21}$$

For $s = 1$ (open state),

$$E(1) = \epsilon_{open} - \tau \Delta A \tag{22}$$

(b) The probabilities of the channel being in the closed (P_0) or the open state (P_1) are:

$$P_0 = \frac{1}{Z} \exp\left(-\frac{\epsilon_{closed}}{k_B T}\right) \tag{23}$$

$$P_1 = \frac{1}{Z} \exp\left(-\frac{(\epsilon_{open} - \tau \Delta A)}{k_B T}\right) \tag{24}$$

(c) The average energy of the two-state system is:

$$\begin{aligned} \langle E \rangle &= P_0 E(0) + P_1 E(1) \\ &= \frac{1}{Z}[\epsilon_{closed} \exp\left(-\frac{\epsilon_{closed}}{k_B T}\right) + (\epsilon_{open} - \tau \Delta A) \exp\left(-\frac{(\epsilon_{open} - \tau \Delta A)}{k_B T}\right)] \end{aligned} \tag{25}$$