ME470: Numerical problem sheet 2 - Two-dimensional Cells

A help session will be held during the exercise session on November 21, and December 5. The final report, including the source code, a detailed explanation of your algorithm, and the calculation results must be submitted by December 8. Late submissions will incur a 20% grade reduction per day, and any submission made 5 days or more after the due date will not be graded.

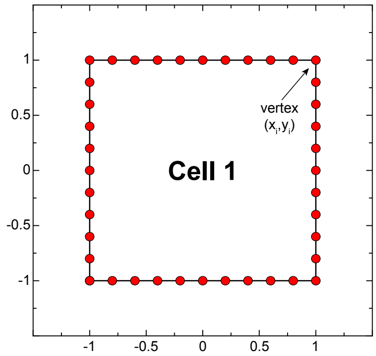

Figure 1: Schematics of the initial cell shape for a single cell

Consider a single cell, whose shape is described by points (vertices) located along the boundary. We will use a square that spans between -1 and 1 on both axes for the initial cell shape. Each edge contains 11 vertices, resulting in a total of 40 vertices for the entire cell (Fig. 1). We will simulate the equilibrium states of this cell under various energy functionals, which correspond to different effective forces acting on the cell.

(1) Foam cell We will first examine a liquid foam cell under constant surface tension, $\gamma = 1$. This tension acts along the edges between adjacent vertices, and each vertex experiences tangential forces from the two neighboring edges.

(a) Generate the initial configuration of a single foam cell as illustrated in Figure. 1. For the data structure, it is useful to define three variables: a vertex variable to store the positions of vertices, an edge variable to store the vertex-edge relation, and a face variable to store an ordered list of vertices.

(b) The foam energy is given by $U_f = \gamma \sum_{i,j} L_{ij}$, where L_{ij} represents the edge length between vertex i and j. This energy reflects a constant tension acting tangentially along each edge. Using these effective forces, the net force on each vertex can calculated. Find the equilibrium configuration under constant tension, and provide reasoning for why this particular equilibrium shape is achieved (To find an equilibrium configuration, you need to numerically iterate adjustment of particle positions in the direction of net force until the maximum net force for vertices is less than certain criterion, $\epsilon_c = ‘0^{-8}$. You need to identify all forces acting on each vertex, compute net force using vector sum, and displace vertex positions in the force direction with a small step, $dt = 0.01$).

(c) Introduce pressure force for a given cell. Assuming that a cell has an equilibrium area, A_0, the normal force on each edge is proportional to the pressure, $K_A(A - A_0)$, and the interfacial length, L_{ij}. While this force is distributed along the edge, it can be replaced as two-point forces on two end vertices. Thus, each vertex now experiences two tangential forces and two normal forces. Adjust A_0 between 0.5 to 5, keeping $K_A = 10$, and find the equilibrium configuration.

(d) Show that the result from part (c) agrees with the Young-Laplace equation.

(2) Elastic cell Now, consider a single cell with perimeter elasticity, where the perimeter behaves like a spring with a fixed length P_0 and energy functional $U_p = \frac{1}{2}K_P\left(P_i - P_0\right)^2$. This can be interpreted as an effective tension along the cell boundary, with a magnitude $K_P\left(P_i - P_0\right)$. Set $K_P = 1$ for all following simulations.

(a) Find the equilibrium configuration for P_0 values varying from 0.5 to 5, using the given initial configuration and excluding the pressure term. Provide a qualitative explanation of the equilibrium configurations for different values of P_0.

(b) Find the equilibrium configuration for P_0 values varying from 0.5 to 5, with the pressure term, $A_0 = 1$ and $K_A = 10$. Do the equilibrium configurations exhibit any qualitative changes as P_0 increases? If so, explain the reason behind this transition in cell shape and estimate the critical value of P_0 at which the change occurs.

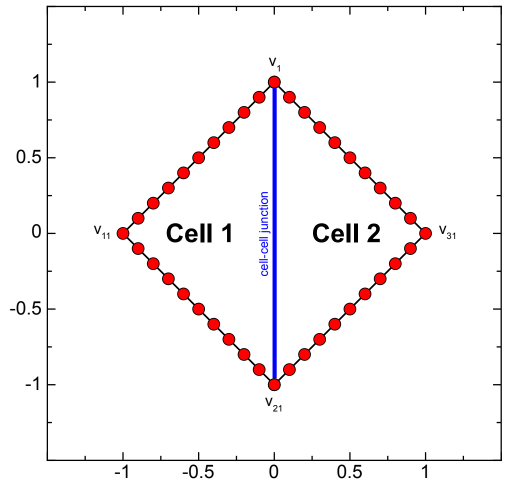

Figure 2: Schematics of the initial cell shape for adherent cells

(3) Cell-cell adhesion Now, consider two cells, sharing a cell-cell contact with adhesion strength W, and cortical tension T_0. These cells can be modeled as liquid foam cells, with an effective tension of $2T_0 - W$ at the cell-cell contact, and T_0 at the free edges. Using the same pressure term $K_A(A_i - A_0)$, we will simulate how changes in adhesion strength affect the equilibrium cell shapes, starting with the initial configuration of two triangular cells described in Figure. 2. Without loss of generality, T_0 is set to 1, and the range of W is between 0 to 2. The relevant parameters for the pressure term are fixed as $K_A = 10$ and $A_0 = 1$.

(a) Find the equilibrium configurations for W varying from 0 to 2, and plot the normalized cell-cell contact length, $L/\sqrt{A_0}$ in terms of W/T_0.

(b) Explain the equilibrium configurations for different values of W using tension balance at each vertex. Analytically compute the equilibrium configurations by assuming (1) a straight edge at the cell-cell junction, (2) a circular arc shape for the free edge, and (3) a fixed area of $A_0 = 1$. Compare the results of the analytic calculation with the simulations.