ME 470: Mechanics of Soft and biological matter.

LECTURE 6: 2D NETWORK - REGULAR TRIANGULAR LATTICE

CONTENTS

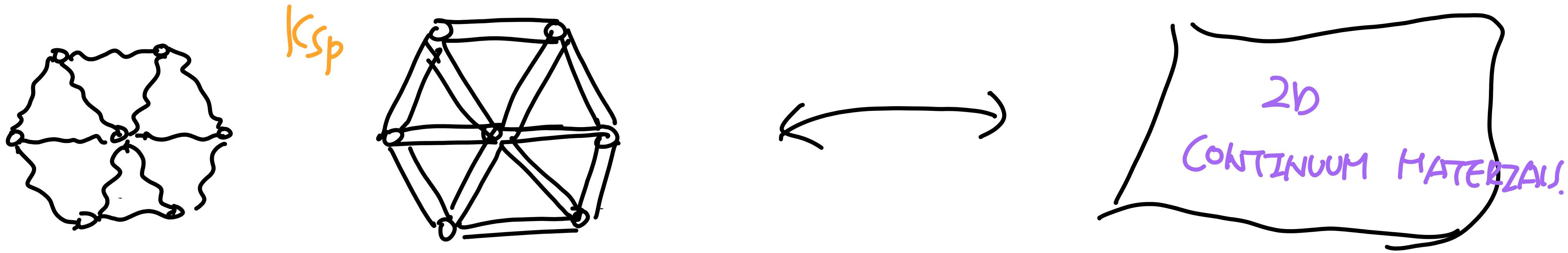
- 2D ELASTIC MODULUS OF REGULAR TRIANGULAR LATTICE
- REGULAR TRIANGULAR LATTICE.

3.3. STRETCHING IN SPRING NETWORK.

FOR A GENERAL NETWORK STRUCTURE, MECHANICAL PROPERTIES DEPEND ON MULTIPLE FACTORS SUCH AS.

1. MECHANICAL PROPERTIES OF A SINGLE 1D ELEMENT
2. ORIENTION OF 1D ELEMENT
3. NUMBER OF 1D ELEMENTS IN A UNIT AREA
4. CROSS-LINK BETWEEN 1D ELEMENTS.

HERE, WE CONSIDER A VERY SIMPLE CASE.



LINEAR SPRINGS ELASTIC STRIPS

INTERPRET REGULAR TRIANGULAL SPRING NETWORK AS 2D LINEAR ELASTIC MATERIALS.

$\Rightarrow$ CONSTITUTIVE RELATION: $\tau_{ij} = C_{ijkl}\,\varepsilon_{kl}\ , \quad i,j,k,l \in \{x,y\}$

NOT 16 UNIQUE COEFFICIENTS.

SYMMETRIES. $C_{ijkl} = C_{jikl} = C_{ijlk}$

THERMODYNAMICS $C_{ijkl} = C_{klij}$

$\Rightarrow$ LEAVES ONLY $C_{xxxx},\ C_{xxxy},\ C_{xxyy},\ C_{xyxy},\ C_{yyyy}.\ C_{yyyx}$

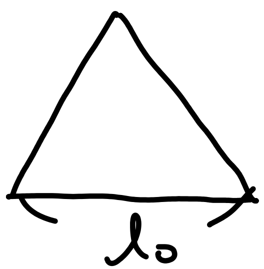
$\Rightarrow$ 6-FOLD SYMMETRY ONLY LEAVES 2 INDEPENDENT COEFFICIENTS.

TRIANGULAR SPRING NETWORK AS 2D ISOTROPIC HOMOGENEOUS ELASTIC MATERIALS.!

3.3.1. BULK MODULUS.

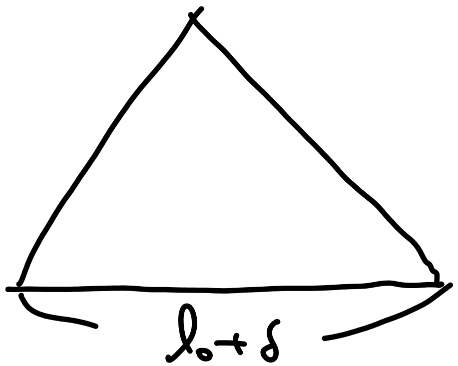
DUE TO SYMMETRY, WE ONLY NEED TO CONSIDER A SINGLE TRIANGLE UNIT, "PLAQUETTE"

$\Rightarrow$ UNDER EQUIBIAXIAL EXTENSION:



$$a_0 = \frac{\sqrt{3}}{4} l_0^2$$

$$\longrightarrow$$



$$a = \frac{\sqrt{3}}{4} (l_0+\delta)^2 \simeq \frac{\sqrt{3}}{4} l_0^2 + O(\delta)$$

$\Rightarrow$ ENERGY PER PLAQUETTE: 2 PLAQUETTE SHARE 1 SPRING.

$$U = 3 \cdot \frac{1}{2} k_{sp} \delta^2 \cdot \left(\frac{1}{2}\right) = \frac{3}{4} k_{sp} \cdot \delta^2$$

3 SPRINGS PER PLAQUETTE

ENERGY PER AREA:

$$u_A = \frac{U}{a_0} = \sqrt{3}\, k_{sp} \left(\frac{\delta}{l_0}\right)^2 \quad \text{—— ①}$$

STRAIN ENERGY DENSITY:

$$\left[\ \varepsilon_{ll}? \qquad \varepsilon_{xx} = \varepsilon_{yy} = \frac{\delta}{l_0}, \qquad \varepsilon_{xy} = 0\ \right]$$

$$u_A = \mu_A \left[2\varepsilon_{xy}^2 + \frac{1}{2}(\varepsilon_{xx} - \varepsilon_{yy})^2 \right] + \frac{K_A}{2}(\varepsilon_{xx} + \varepsilon_{yy})^2$$

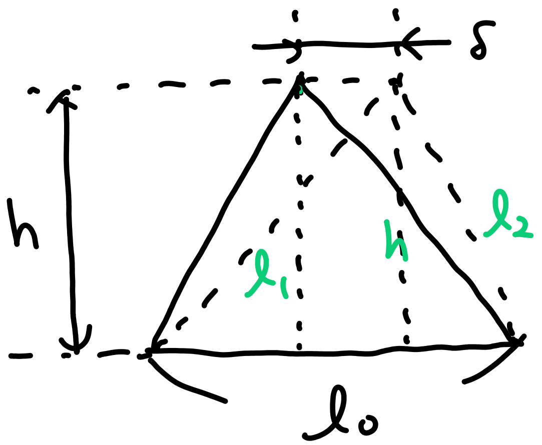
$$\Rightarrow \quad u_A = \frac{K_A}{2}(\varepsilon_{xx} + \varepsilon_{yy})^2 = 2K_A \cdot \left(\frac{\delta}{l_0}\right)^2 \quad \text{— ②}$$

EQUIBIAXIAL EXTENSION.

① = ② $\Rightarrow$ $\boxed{K_A = \frac{\sqrt{3}}{2} k_{sp}}$

3.3.2. SHEAR MODULUS.

SIMPLE SHEAR



$$l_1^2 = \left(\frac{l_0}{2}+\delta\right)^2 + \overbrace{\frac{3}{4}l_0^2}^{h^2}$$

AND STILL $a = a_0 = \frac{\sqrt{3}}{4}l_0^2$

(FOR SMALL δ) $\Rightarrow$

$$l_1 \simeq l_0 + \frac{\delta}{2}$$

$$l_2 \simeq l_0 - \frac{\delta}{2}$$

ENERGY PER PLAQUETTE

$$U = 2\cdot\frac{1}{2}k_{sp}\cdot\left(\frac{\delta}{2}\right)^2\cdot\frac{1}{2} = \frac{k_{sp}}{8}\delta^2$$

$$\Rightarrow u_A = \frac{U}{a_0} = \frac{k_{sp}}{2\sqrt{3}}\cdot\left(\frac{\delta}{l_0}\right)^2 \quad \text{—— ③}$$

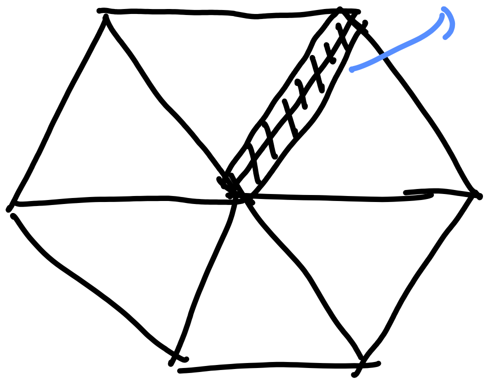
STRAIN ENERGY DENSITY

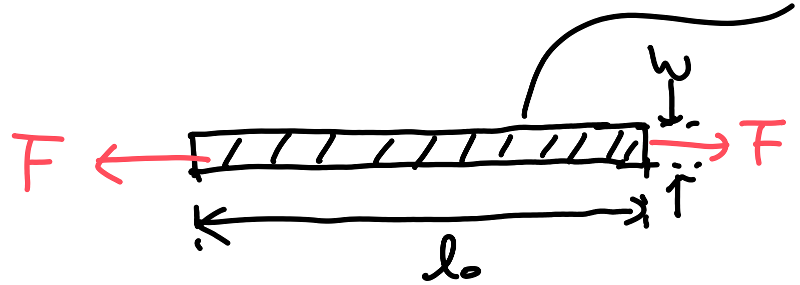
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \cancelto{0}{\frac{\partial u_y}{\partial x}}\right) = \frac{1}{2}\cdot\frac{\delta}{h} = \frac{\delta}{\sqrt{3}\,l_0}$$

$$u_A = \mu_A\cdot 2\cdot\left(\frac{\delta}{\sqrt{3}\,l_0}\right)^2 = \frac{2}{3}\mu_A\left(\frac{\delta}{l_0}\right)^2 \quad \text{—— ④}$$

$$\boxed{\text{③} = \text{④ YIELD} \quad \mu_A = \frac{\sqrt{3}}{4}k_{sp} = \frac{1}{2}K_A}$$

3.3.3. ELASTIC STRIPS





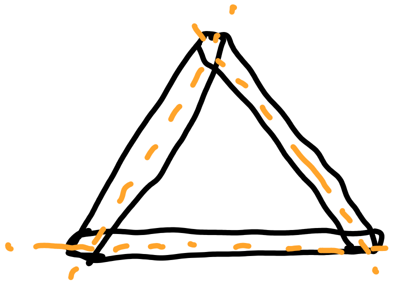
, $F = k_{sp} \cdot \delta$

UNIAXIAL TENSION: $\tau_{11} = E_A^{MAT} \cdot \varepsilon_{11}$

(WE ALREADY COMPUTED $E_A^{MAT} = 2\, K_A^{MAT} \cdot (1 - \nu_A^{MAT})$)

$$\tau_{11} = \frac{F}{w} = k_{sp} \cdot \left(\frac{\delta}{w}\right) = k_{sp} \cdot \left(\frac{l_0}{w}\right) \cdot \underbrace{\left(\frac{\delta}{l_0}\right)}_{\varepsilon_{11}}$$

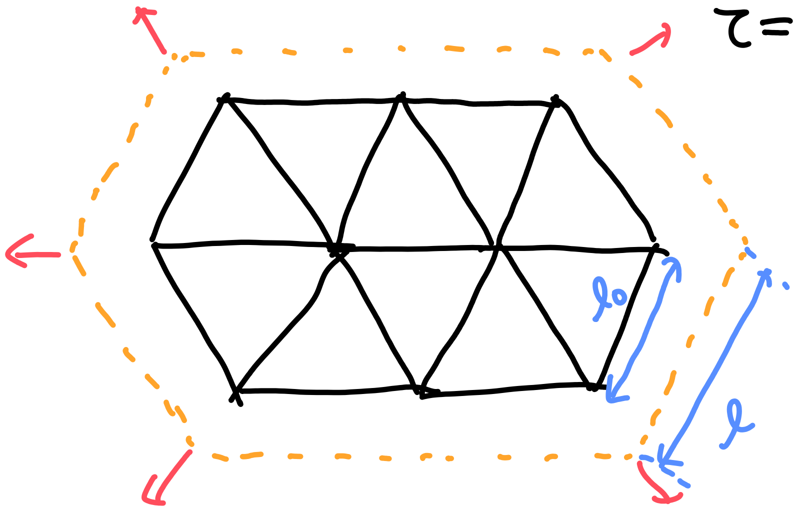
$$\Rightarrow E_A^{MAT} = k_{sp} \cdot \left(\frac{l_0}{w}\right) = \frac{2}{\sqrt{3}} K_A \cdot \left(\frac{l_0}{w}\right) = 2\, K_A^{MAT} (1 - \nu_A^{MAT}) \Rightarrow \boxed{K_A = \sqrt{3} \cdot \frac{w}{l_0} \cdot (1 - \nu_A^{MAT}) K_A^{MAT}}$$



AREA FRACTION OF STRIPS?

$$\varphi = \frac{3 \cdot w \cdot l_0}{\frac{\sqrt{3}}{4} l_0^2} \cdot \left(\frac{1}{2}\right) = 2\sqrt{3} \cdot \frac{w}{l_0} \Rightarrow \boxed{K_A = \frac{\varphi}{2} \cdot (1 - \nu_A^{MAT}) K_A^{MAT}}$$

3.4. NETWORK UNDER EXTERNAL STRESS



$\tau = \text{CONST}$

SIMPLEST: $\tau > 0$

PER PLAQUETTE: $U = \frac{3}{4} k_{sp}\, \delta^2$ $\quad (\delta = l - l_0)$

$$u_A = \sqrt{3}\, k_{sp} \left(\frac{\delta}{l_0}\right)^2$$

CONSIDER ENTHALPY $\quad h_A = u_A - \tau_{ij}\varepsilon_{ij}$

$$H = U - \int_{\text{PLAQUETTE}} \tau_{ij}\varepsilon_{ij}\, dA$$

HERE: $\quad \tau_{ij}\varepsilon_{ij} = \tau \cdot (\varepsilon_{xx} + \varepsilon_{yy}) = \tau \cdot \frac{\Delta a}{a_0} \qquad (\Delta a = a - a_0)$

$$\Rightarrow H = U - \tau \Delta a = U - \tau a + \cancel{\tau a_0}$$

CONSTANT, NEGLECT

$$H = \frac{3}{4} k_{sp} (l - l_0)^2 - \tau \cdot \frac{\sqrt{3}}{4} l^2$$

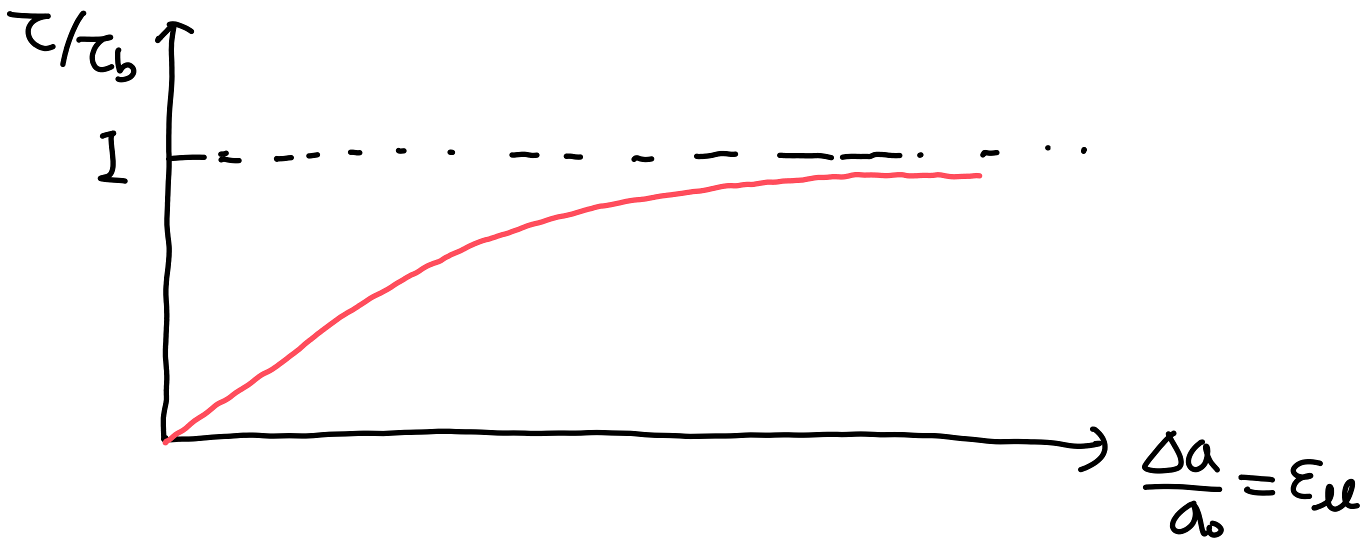
EQUILIBRIUM: $dH=0 \iff \frac{dH}{dl}=0$ (ALL EQUILATERAL $\triangle$ ONLY, 1 VARIABLE)

$$\frac{3}{2}k_{sp}(l-l_0)=\tau\cdot\frac{\sqrt{3}}{2}l$$

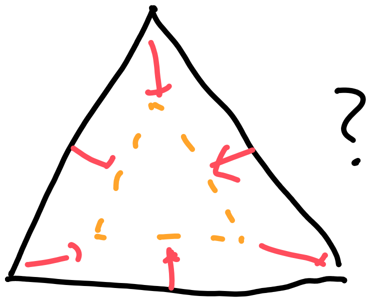
$$\Rightarrow \boxed{l=\frac{l_0}{1-\tau/\tau_b}}$$

WHERE $\boxed{\tau_b=\sqrt{3}\,k_{sp}}$

CATASTROPHIC BLOW-UP $(l\to\infty)$ AS $\tau\to\tau_b$



WHAT IF $\tau < 0$ (STILL UNIFORM)?



BUT, DEFORMATION CAN BE NON-UNIFORM.

$\Rightarrow$ HUGE # OF VARIABLES l_i IN H

$\Rightarrow$ NEED ALL $\dfrac{\partial H}{\partial l_i} = 0 \quad \forall i$

TO SOLVE THIS PROBLEM..

(1) NUMERICAL SOLUTION.

(2) CALCULUS OF VARIATION.

(3) MODE ANALYSIS

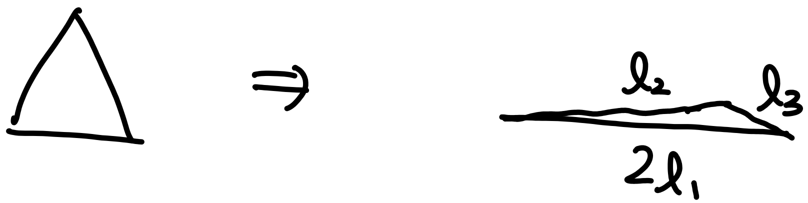
$\hookrightarrow$ PICK DISTINCTIVE MODES OF DEFORMATION, COMPUTE ENERGY FUNCTIONAL VALUES.

MODE 1 UNIFORM CONTRACTION.

$$H_u = \frac{3}{4} k_{sp} (l - l_0)^2 - \tau \cdot \frac{\sqrt{3}}{4} l^2$$ (SAME AS EXTENSION)

$$H_u^{eq}(\tau) = \frac{3}{4} k_{sp} l_0^2 \left(1 - \frac{1}{1 - \tau/\tau_b}\right) - \tau \cdot \frac{\sqrt{3}}{4} \cdot \frac{l_0^2}{(1 - \tau/\tau_b)^2}$$

MODE 2 PLAQUETTE COLLAPSE $(a \to 0)$



$$2l_1 = l_2 + l_3$$

ASSUMPTION: ALL PLAQUETTES COLLAPSE THE SAME WAY

$$H_c = U - \cancel{\tau a}^{\,0} = \frac{1}{2} k_{sp} \left[(2l_1 - l_0)^2 + (l_2 - l_0)^2 + (2l_1 - l_2 - l_0)^2 \right] \cdot \frac{1}{2}$$

$$dH_c = 0 \iff \frac{\partial H_c}{\partial l_1} = 0 \quad \text{AND} \quad \frac{\partial H_c}{\partial l_2} = 0$$

$$\frac{\partial H_c}{\partial l_2} = \frac{1}{4}k_{sp}\left[\,2(l_2-\cancel{l_0}) - 2(2l_1-l_2-\cancel{l_0})\right]=0$$

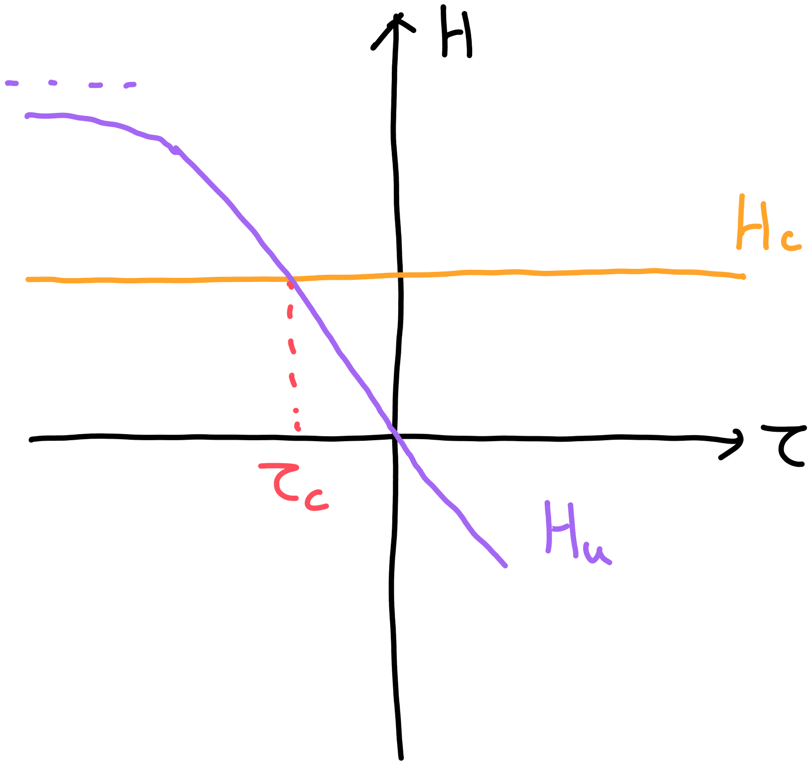
$$\frac{k_{sp}}{4}\left[\,4l_2-4l_1\right]=0 \qquad \Rightarrow \qquad \boxed{l_1=l_2}$$

$$\Rightarrow \quad H_c = \frac{k_{sp}}{4}\left[(2l_1-l_0)^2 + 2(l_1-l_0)^2\right]$$

$$\frac{\partial H_c}{\partial l_1} = \frac{k_{sp}}{4}\left[2\cdot(2l_1-l_0)\cdot 2 + 4(l_1-l_0)\right]$$

$$= k_{sp}\left[\,3l_1-2l_0\right] \overset{!}{=} 0 \qquad \Rightarrow \quad l_1 = \frac{2}{3}l_0$$

$$\Rightarrow \quad H_c^{eq} = \frac{k_{sp}}{4}\left[\frac{l_0^2}{9} + 2\cdot\frac{l_0^2}{9}\right] \doteq \boxed{\frac{1}{12}k_{sp}l_0^2}$$



$$H_c^{eq} = H_u^{eq}$$

$$\Rightarrow \frac{1}{12} k_{sp} \cancel{l_0^2} = \frac{3}{4} k_{sp} \cancel{l_0^2} \left(1 - \frac{1}{1-\tau/\tau_b}\right)^2 - \tau \cdot \frac{\sqrt{3}}{4} \cdot \frac{\cancel{l_0^2}}{(1-\tau/\tau_b)^2}$$

$x = \tau/\tau_b$

$$1 = 9\left(1 - \frac{1}{1-\tau/\tau_b}\right)^2 - \frac{9\tau}{\underbrace{\sqrt{3}\, k_{sp}}_{\tau_b}} \cdot \frac{1}{(1-\tau/\tau_b)^2}$$

$$1 = 9 \cdot \left(\frac{-x}{1-x}\right)^2 - \frac{9x}{(1-x)^2}$$

$$(1-x)^2 = 9x^2 - 9x$$

$$8x^2 - 7x - 1 = 0$$

$$(8x+1)(x-1) = 0 \quad \Rightarrow \quad x = \begin{cases} -1/8 \\ 1 \end{cases}$$

$$\boxed{\tau_c = -\frac{1}{8}\tau_b}$$

NOTE:

$$l(\tau_c) = \frac{l_0}{1 - \tau_c/\tau_b} = \frac{8}{9} l_0$$

(STILL UNIFORM)

NOTE:

(1) ONLY 2 MODES ANALYZED, OTHERS MAY HAVE $H < H_u$ FOR $\tau > \tau_c$, AND LEAD TO INSTABILITY FIRST.

(2) NETWORK MUST DEFORM FROM U TO C THROUGH INTERMEDIATE STATES, THESE MIGHT HAVE HIGHER H^*

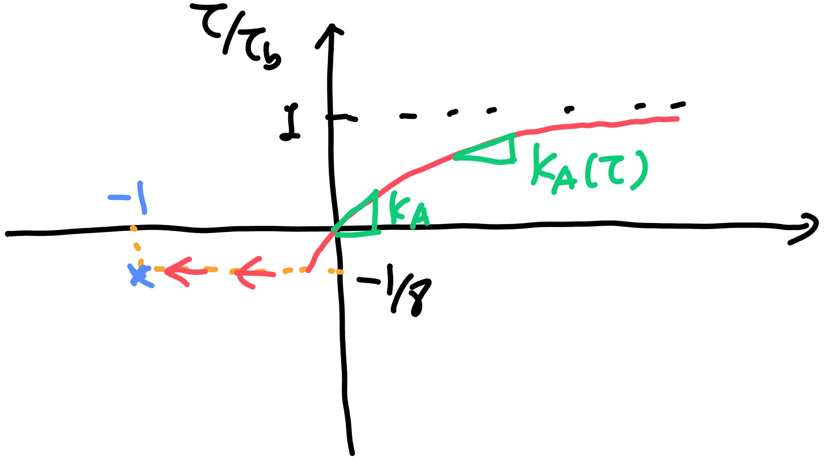
SUMMARY) REGULAR TRIANGULAR NETWORK.

$$\left.\begin{aligned} K_A &= \frac{\sqrt{3}}{2} k_{sp} \\ \mu_A &= \frac{\sqrt{3}}{4} k_{sp} \end{aligned}\right\} \text{LINEAR.}$$

ENERGY ANALYSIS. (UNIFORM TENSILE STRESS, $\tau > 0$)

$$l(\tau) = \frac{l_0}{1 - \tau/\tau_b}, \qquad \tau_b = \sqrt{3}\, k_{sp}$$

MODE ANALYSIS $(\tau < 0)$ $\quad H_c < H_u$ FOR $\tau < -\frac{1}{8}\tau_b$



$$\varepsilon_{ll} = \frac{\Delta a}{a_0} = \frac{a - a_0}{a_0}$$