Generic Rigidity Percolation: The Pebble Game

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The percolation of rigidity in 2D central-force networks with no special symmetries (generic networks) has been studied using a new combinatorial algorithm. We count the exact number of floppy modes, uniquely decompose the network into rigid clusters, and determine all overconstrained regions. With this information we have found that, for the generic triangular lattice with random bond dilution, the transition from rigid to floppy occurs at $p_{cen} = 0.6602 \pm 0.0003$ and the critical exponents include $\nu = 1.21 \pm 0.06$ and $\beta = 0.18 \pm 0.02$.

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The percolation of rigidity on random central-force networks has been studied over the past 12 years [1–10]. One of the most interesting findings has been that effective medium theory describes the behavior of the elastic constants and the number of floppy modes remarkably well [11,12], except very close to the phase transition from rigid to floppy. The success of effective medium theory has allowed complex situations, such as the floppy modes and elastic behavior of 3D glasses like $Ge_xAs_ySe_{1-y}$ to be well characterized [1,13]. However, attempts to study the critical behavior of central-force networks have not been very satisfactory, and the question of the universality class [5,6,8–10] remains unresolved. This question is fundamental in understanding the nature of the rigidity transition, and may have important implications as to how the character of the glass transition is affected by the mean coordination, as has been discussed recently via *fragile* and *strong* glass formers [14].

In this Letter, we point out that many of these difficulties arise because the *wrong problem* has been studied. Using concepts from graph theory, we set up *generic networks* that are the set of all networks with a given topology and *no* special symmetries. Real glasses are well represented by generic networks. In generic networks all infinitesimal floppy motions carry over to finite motions [7,15]. A simple way to construct a generic network is to take a regular lattice structure and randomly displace each site location by a small amount, thereby eliminating the presence of connected collinear bonds and parallel bonds. Instead of these special geometries, we show in Figs. 1(a) and 1(b) their generic counterparts.

As a paradigm model we have revisited the problem of rigidity percolation on a bond-diluted triangular network with central forces [2,11] but which is constructed to be generic. Hence the topology is that of a triangular lattice, but the bond lengths and bond angles are locally distorted. This eliminates the *diode* problem where a pair of collinear bonds can only be extended with a cost in energy, but can be compressed with no cost in energy due to buckling [see Fig. 1(a)]. The diode effect complicates studies because it leads to the breakdown of linear elasticity, which should be reversible. Therefore, prior studies on the nonlinear effects arising from geometrical singularities [7] should be considered as a separate problem. Here we give the first answers regarding the geometric nature of rigidity percolation on generic networks and establish a framework for further discussion.

We focus on the geometrical aspects of rigidity percolation that have not been directly addressed before. Previous studies [3–5,9] have used costly relaxation methods where networks containing 10^4 sites already present a difficult numerical challenge. It is now possible to study networks containing up to 10^7 sites, using integer algorithms, which give exact and unique answers to the geometric properties of rigidity percolation. This is the first paper on generic rigidity percolation, and many more studies

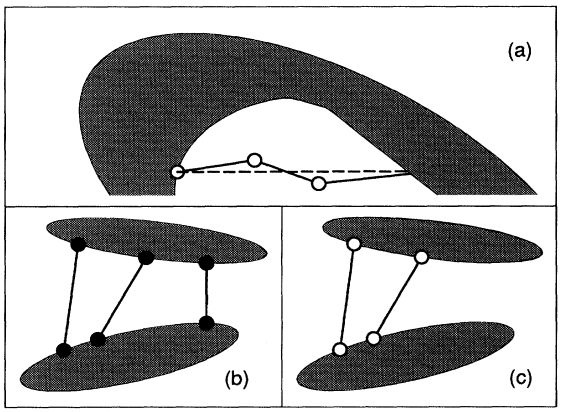

FIG. 1. The shaded regions represent 2D rigid bodies. The (closed, open) circles denote pivot joints that are members of (one, more than one) rigid body. (a) Three noncollinear rods connecting across a rigid body are generic with one internal floppy mode. Collinear rods along the dotted line would yield two *infinitesimal* floppy modes, but buckling can occur. (b) Three generic bars connecting two rigid bodies form a rigid structure. If the rods were parallel, it would not be rigid to shear [16]. (c) Four distinct rigid clusters.

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and applications are suggested. Here we want to emphasize the conceptual advantage of this new viewpoint.

Much understanding of the general phenomena of central-force rigidity percolation can be obtained by studying a network of bars and joints [1,7]. This is because *rigidity* is a static concept, involving virtual displacements. A collection of sites form a rigid cluster when no relative motion within that cluster can be achieved without a cost in energy. Conversely, the floppy modes correspond to *finite* motions of the sample, which do not cost energy.

The number of floppy modes in d dimensions is given by the total number of degrees of freedom for N sites minus the number of *independent* constraints. A key quantity is the number of floppy modes per degree of freedom, f, in the network. By defining the number of redundant bonds per degree of freedom as n_r, we can write quite generally,

$$f = \frac{dN - (\frac{1}{2}Nzp - dNn_r)}{dN} = 1 - \frac{p}{p^*} + n_r, \quad (1)$$

where $p^* = 2d/z$ and z is the lattice coordination. Neglecting the redundant bonds, as first done by Maxwell [17], we find that f is linear in the bond concentration, p, and goes to zero at the Maxwell approximation p^* for the threshold as shown in Fig. 2(a). The Maxwell approximation gives a very good account of the location of the phase transition and the number of floppy modes, but it ultimately fails since the number of independent constraints is not just the total number of bonds as some bonds are *redundant*.

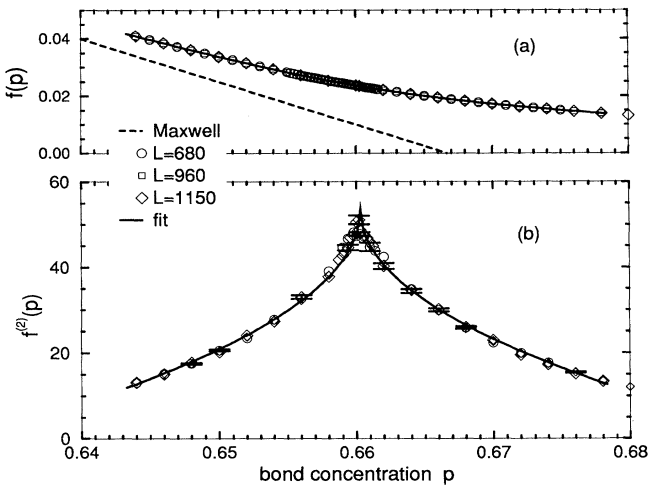

FIG. 2. (a) Simulation results for the fraction of floppy modes, f, compared to the Maxwell prediction. Error bars are smaller than the symbol sizes. (b) A cusp in the second derivative, $f^{(2)}$, is found at 0.6603 ± 0.0003, which locates the transition. Typical error bars are shown on several data points. The solid line in (a) and (b) represent the best fit for f and $f^{(2)}$, respectively. We find a cusp exponent of -0.48 ± 0.05.

There are two important differences between connectivity and rigidity percolation. The first difference is that rigidity percolation is a vector (not a scalar) problem, and secondly, there is an inherent *long range* aspect to rigidity percolation. Figure 1(c) shows four distinct rigid clusters (two rods and two clusters). Now the placement of one additional bond, as in Fig. 1(b), *locks* all four clusters into a single rigid cluster. This nonlocal character allows a single bond on one end of the network to affect the rigidity *all across* the network from one side to the other.

We have implemented an efficient combinatorial algorithm suggested by Hendrickson [15] to (i) calculate the number of floppy modes, (ii) locate overconstrained regions, and (iii) identify all rigid clusters for 2D generic bar-joint networks. The crux of the algorithm is based on a theorem by Laman [18] from graph theory.

Theorem: *A generic network in two dimensions with N sites and B bonds (defining a graph) does not have a redundant bond if no subset of the network containing n sites and b bonds (defining a subgraph) violates $b \leq 2n - 3$.*

By simple constraint counting it can be seen that there must be a redundant bond when Laman's condition is violated. This necessary part generalizes to all dimensions. However, the essence of Laman's theorem is that in two dimensions finding $b > 2n - 3$ is the only way redundant bonds can arise. This sufficient part does not generalize to higher dimensions [15].

The basic structure of an efficient algorithm is to apply Laman's theorem recursively by building the network up one bond at a time. Only subgraphs that contain the newly added bond need to be checked. If each of these subgraphs satisfies the Laman condition, then the last bond placed is independent; otherwise it is redundant.

The algorithm can be understood quite intuitively. Searching over the subgraphs is accomplished by constructing a *pebble game* [19]. A pebble is either *free* when it is on a site or *anchored* when it is covering a bond. A free pebble represents a single motion that a site can undertake. A site with two free pebbles has two translational motions. If two additional free pebbles can be found at a different site, then the distance between these sites is not fixed. Placing a bond between this pair of sites will constrain their distance of separation. This independent constraint is recorded by anchoring one of the four free pebbles to the bond, which must always remain covered.

We begin with a network of $N = L^2$ sites. Nearest neighbor bonds are randomly selected and tested in turn. Then pebbles are shuffled around the network in an attempt to free two pebbles at each site at the ends of the test bond. In the shuffling, an anchored pebble can be released from covering a bond by anchoring a neighboring free pebble to that bond. Figures 3(a) and 3(b) show an example of how pebbles are shuffled. It is always possible to free three pebbles, since they correspond to the rigid body motion of that bond. A redundant bond is identified

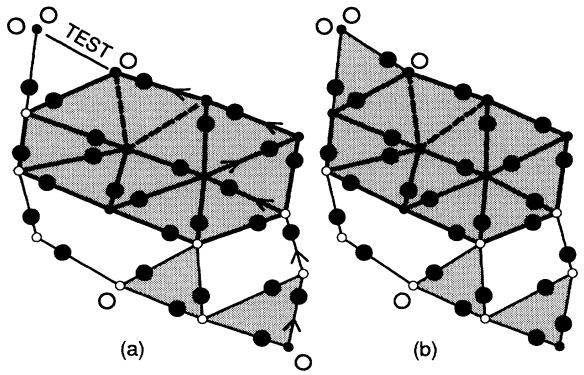

FIG. 3. A demonstration of the pebble game on a generic network. Independent (redundant) bonds are shown with solid (dashed) lines that are (are not) covered by a pebble. Large (filled, open) circles denote (anchored, free) pebbles on (bonds, sites). Small (filled, open) circles denote sites belonging to (one, more than one) rigid cluster. Overconstrained bonds are shown with heavy dark lines. Shaded regions denote 2D rigid bodies. (a) Five free pebbles indicate 5 floppy modes until a new bond is added and tested for independence. A fourth free pebble is found via the path traced by arrows. (b) The added bond is independent and thus covered.

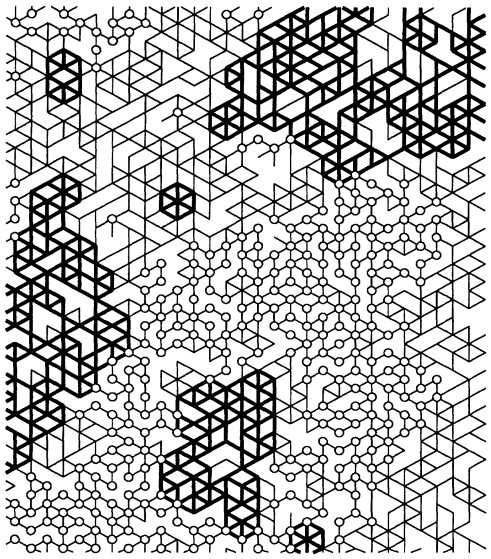

FIG. 4. The *topology* of a typical cutout region from a bond-diluted generic triangular lattice. A particular realization would have local distortions (not shown) similar to Fig. 3. Heavy dark lines denote overconstrained bonds. Open circles denote pivots between two or more rigid bodies.

when the search for the fourth pebble results in a closed loop back to the sites at the ends of the test bond. This indicates that the distance between the incident sites is already fixed, and the redundant bond is not covered.

The exact number of floppy modes is determined by counting the number of redundant bonds. When a redundant bond is found, the set of sites searched in the failed attempt to free the fourth pebble defines an overconstrained region. A systematic search to map out *all* rigid clusters is made *after* building the network. The rigidity between a pair of sites is checked using a test bond. If the test bond is redundant, then the sites are mutually rigid. Figure 4 shows a section of a large network after applying the *pebble game*.

The *pebble game* is slowest at the rigidity threshold where it scales as $\sim N^{1.2}$ to system size. On average it takes $\approx$1.4 CPU minutes on a Dec-alpha workstation for processing a 1150×1150 site system. Away from the threshold, the CPU time scales nearly linearly ($\sim N$).

A sum rule for f in terms of the rigid clusters can be written as

$$f(p) = 1 - \frac{1}{2}\sum_{s=2}^{N} n_s(p)(2s - 3), \qquad (2)$$

where n_s is the number of rigid clusters with s sites per lattice site. Unlike connectivity percolation, we have $\sum_{s=1}^{N} s n_s(p) \geq 1$ because sites are shared at pivots. For bond dilution, it can be shown that $f^{(1)} = df/dp = -\frac{3}{2}(1 - N_O/N_B)$, where N_O is the number of bonds forming the overconstrained regions and N_B is the total number of bonds in the network.

The second derivative, $f^{(2)}$, can be calculated by sampling the change in N_O with the random placement of one additional bond. In Fig. 2(b) we plot $f^{(2)}$ versus bond concentration, p, and find a sharp peak at the threshold. The behavior of $f^{(2)}$ suggests that the number of floppy modes is analogous for rigidity and connectivity percolation. In the latter case, the number of floppy modes corresponds to a *free energy* [20]. Assuming f corresponds to a free energy density, we fit our data for f, $f^{(1)}$, and $f^{(2)}$ to obtain the cusp exponent $\alpha = 0.48 \pm 0.05$ and a threshold of $p_{\text{cen}} = 0.6603 \pm 0.0003$. We note that the total number of rigid clusters is *not* suitable as a free energy because its second derivative changes sign at the transition and thus violates convexity requirements.

We also analyzed our rigid cluster data using finite size scaling where a single relevant length scale is assumed [21]. Each network was filled to a predetermined *exact* bond concentration. We present results from data taken from the set of linear system sizes $L = \{25, 35, 50, 70, 100, 120, 170, 240, 340, 480, 680, 960\}$ and have generated N_R realizations (with $L^2 N_R \geq 10^8$ for $L \geq 100$ and $N_R = 10\,000$ for $L < 100$) per fixed concentration p for both periodic and free boundary conditions. This extensive simulation is made possible by considering *generic* networks.

We define a network as percolating when a spanning rigid cluster exists. The width in the probability density for a network of linear size L to percolate should scale as $\Delta \sim AL^{-1/\nu}$. Estimating $\Delta(L)$ from $\sqrt{\langle p^2 \rangle - \langle p \rangle^2}$ and incorporating corrections to finite size scaling [22],

the correlation length exponent is extrapolated to be $\nu = 1.21 \pm 0.06$. The first moment, $\langle p \rangle$, is extrapolated to obtain the threshold $p_{\text{cen}} = 0.6602 \pm 0.0003$, which agrees well with the cusp location in $f^{(2)}$. Previous best estimates [5,9] for p_{cen} are 0.641 ± 0.001 for the regular (nongeneric) triangular network. Clearly there are important differences between the generic and nongeneric cases.

The fractal dimension $d_f = d - \beta/\nu$ was obtained using "mass" scaling at $p = p_{\text{cen}}$ by averaging over approximately 4000 spanning clusters at each system size mentioned above in addition to another size of $L = 1150$ averaging over roughly 2000 spanning clusters. Incorporating corrections to finite size scaling [22] we extrapolate $d_f = 1.86 \pm 0.02$. As suggested by Duxbury [23] we examined the fractal dimension of the so-called backbone [3,5,9], which is the stress carrying part of the incipient infinite cluster. The backbone dimension was extrapolated to be 1.80 ± 0.03, which agrees well with Moukarzel and Duxbury [23] as reported in the following Letter.

We estimate from the order parameter, P_∞, defined as the probability for a bond to belong to the incipient infinite cluster, the exponent $\beta = 0.18 \pm 0.02$. Recently, it has been suggested that the transition we see should be first order [10], and it has been shown that the rigidity transition is first order on various Bethe lattices [23]. However, we have not detected any indications of a discontinuity in quantities such as P_∞ or $f^{(2)}$.

We suggest that the rigidity transition is second order for the bond-diluted triangular lattice. We have obtained the exponents $\nu = 1.21 \pm 0.06$ and $\beta = 0.18 \pm 0.02$ using the usual cluster moment definitions [21] except for the "specific heat" exponent. We find $\alpha = 0.48 \pm 0.05$ from $f^{(2)}$, as shown in Fig. 2(b), assuming f is the free energy density. Interestingly, the hyperscaling relation, $\alpha = 2 - d\nu$, is satisfied with our independent estimates for α and ν within the error bars. Further work is needed to establish f as an appropriate free energy.

In summary, we have shown how the concept of a generic graph, as introduced by Laman [18] and amplified by Hendrickson [15] can greatly simplify problems concerned with the percolation of rigidity. The surprise is that networks that lack any symmetry (generic networks) are much easier to deal with. This concept leads to a common p_{cen} for all generic bond-diluted triangular networks. Moreover, the same static critical exponents (e.g., α, β, γ, ν) would be expected to be universal for all 2D generic networks. It is now clear that rigidity percolation (with exponents $\nu = 1.21 \pm 0.06$ and $\beta = 0.18 \pm 0.02$) is in a different universality class than connectivity percolation (with exponents $\nu = 4/3$ and $\beta = 5/36 \approx 0.14$).

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