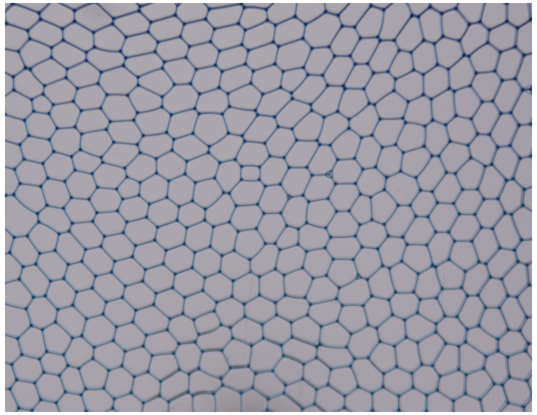
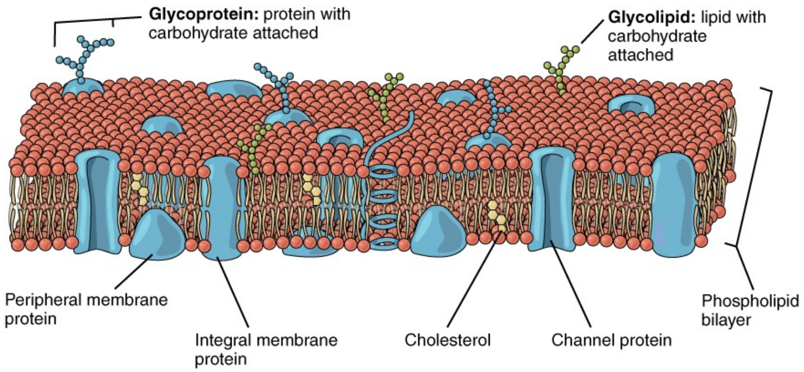
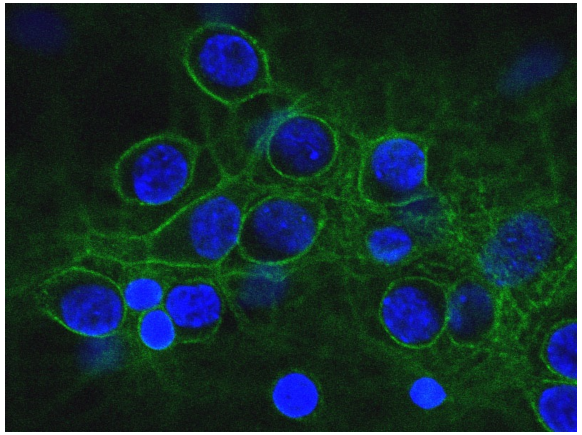
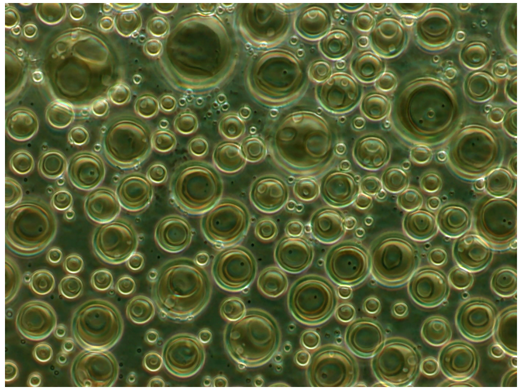
ME470: Mechanics of Soft and Biological Matter
Lecture6: Triangular Lattice









Red Blood Cells

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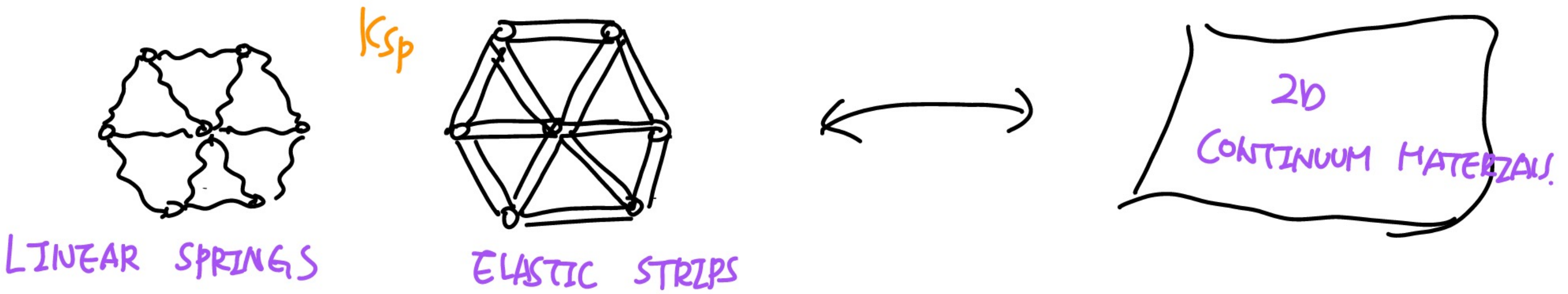
MESOBIO – IGM – STI – EPFL

Stretching in Spring Network

Mechanical properties of general network structures depend on multiple factors:

- Mechanical properties of a single 1D element
- Orientation of 1D element
- Number of 1D elements in a unit area
- Cross-link between 1D elements

We con sider a very simple case:



Constitutive relation (2D linear elastic): $\tau_{ij} = C_{ijkl}\varepsilon_{kl} \qquad i, j, k, l \in \{x, y\}$

⇨ Symmetry and thermodynamics reduces coefficients from 16 to 6

⇨ 6-fold symmetry only leaves 2 independent coefficients

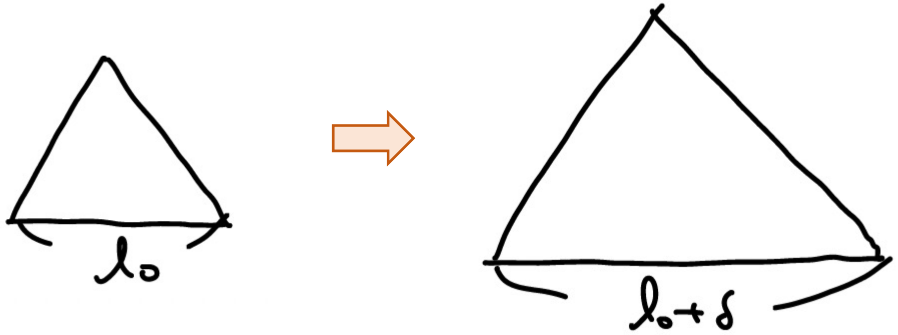
⇨ Regular triangular spring network as 2D isotropic homogeneous elastic materials!

Bulk Modulus

Due to symmetry, only consider a single triangle, “plaquette”

Under equi-biaxial extension:

$$a_0 = \frac{\sqrt{3}}{4} l_0^2$$



$$a = \frac{\sqrt{3}}{4}(l_0+\delta)^2 \approx \frac{\sqrt{3}}{4} l_0^2 + O(\delta)$$

Energy per plaquette:

$$U = 3 \cdot \frac{1}{2} k_{sp} \delta^2 \cdot \left(\frac{1}{2}\right) = \frac{3}{4} k_{sp} \delta^2$$

Bulk Modulus

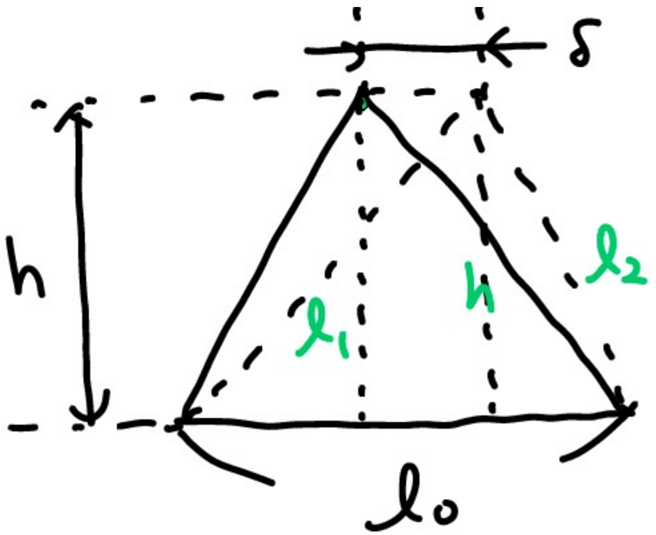
Energy per area: $u_A = \frac{U}{a_0} = \sqrt{3}k_{sp}\left(\frac{\delta}{l_0}\right)^2$

Strain energy density: ε_{ll}? $\varepsilon_{xx} = \varepsilon_{yy} = \frac{\delta}{l_0}$ $\varepsilon_{xy} = 0$

$$u_A = \mu_A\left[2\varepsilon_{xy}{}^2 + \frac{1}{2}\left(\varepsilon_{xx} - \varepsilon_{yy}\right)^2\right] + \frac{K_A}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right)^2 = 2K_A\left(\frac{\delta}{l_0}\right)^2$$

$$K_A = \frac{\sqrt{3}}{2}k_{sp}$$

Shear Modulus



$$l_1^2 = \left(\frac{l_0}{2} + \delta\right)^2 + h^2 \qquad a = a_0 = \frac{\sqrt{3}}{4} l_0^2 \qquad h = \frac{\sqrt{3}}{2} l_0$$

For small δ $\quad l_1 \approx l_0 + \frac{\delta}{2} \qquad l_2 \approx l_0 - \frac{\delta}{2}$

⇨ Energy per plaquette:

$$U = 2 \cdot \frac{1}{2} k_{sp} \left(\frac{\delta}{2}\right)^2 \cdot \left(\frac{1}{2}\right) = \frac{1}{8} k_{sp} \delta^2 \qquad u_A = \frac{k_{sp}}{2\sqrt{3}} \left(\frac{\delta}{l_0}\right)^2$$

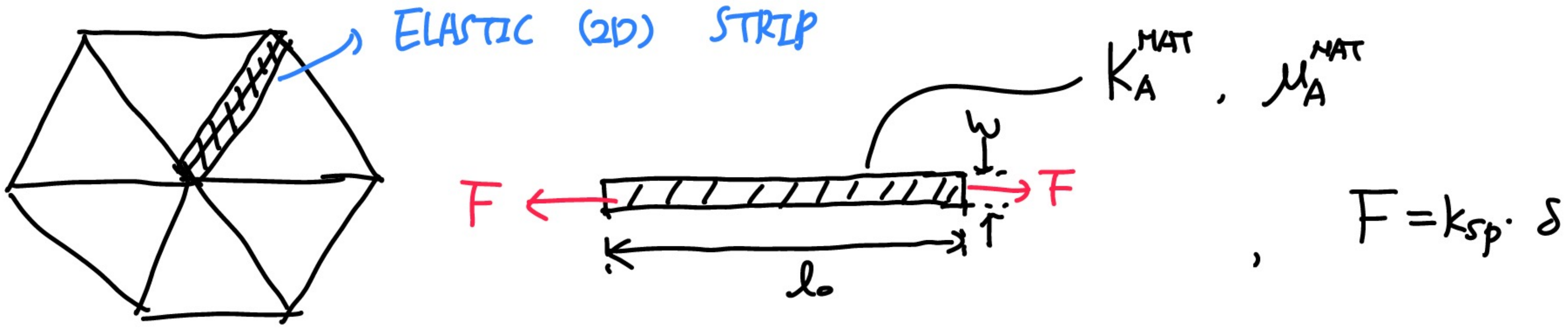
⇨ Strain energy density: $\quad u_A = \mu_A \cdot 2\varepsilon_{xy} = \frac{2}{3} \mu_A \left(\frac{\delta}{l_0}\right)^2$

⇨

$$\mu_A = \frac{\sqrt{3}}{4} k_{sp}$$

EPFL

Elastic strips



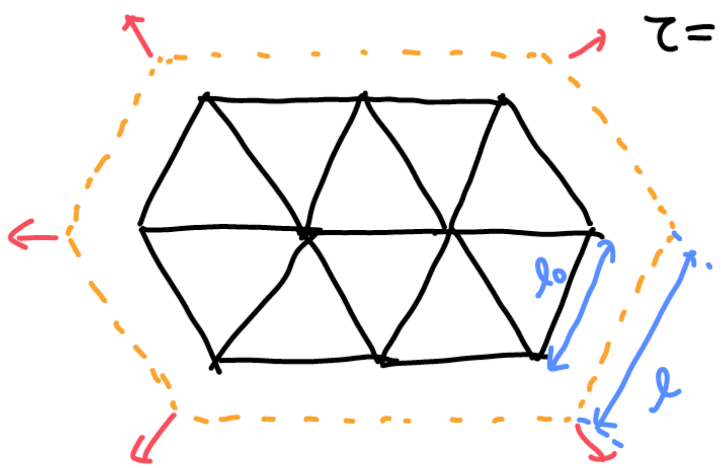
Uniaxial tension: $\tau_{11} = E_A^{MAT} \cdot \varepsilon_{11} = k_{sp}\left(\frac{l_0}{w}\right)\left(\frac{\delta}{l_0}\right)$

$$\Rightarrow \quad E_A^{MAT} = k_{sp}\left(\frac{l_0}{w}\right) = \frac{2}{\sqrt{3}} K_A \left(\frac{l_0}{w}\right) = 2K_A^{MAT}\left(1 - \nu_A^{MAT}\right) \quad \Rightarrow \quad K_A = \sqrt{3}\frac{w}{l_0}\left(1 - \nu_A^{MAT}\right)K_A^{MAT}$$

Area fraction of strip: $\phi = 2\sqrt{3}\frac{w}{l_0}$

$$\Rightarrow \quad K_A = \frac{\phi}{2}\left(1 - \nu_A^{MAT}\right)K_A^{MAT}$$

Network under External Stress



Per plaquette: $U = \frac{3}{4} k_{sp} \delta^2$ $(\delta = l - l_0)$

$$u_A = \sqrt{3} k_{sp} \left(\frac{\delta}{l_0}\right)^2$$

Enthalpy: $h_A = u_A - \tau_{ij}\varepsilon_{ij}$

$$H = U - \int_{Plaquette} \tau_{ij}\varepsilon_{ij}\, dA$$

⇨ $\tau_{ij}\varepsilon_{ij} = \tau(\varepsilon_{xx} + \varepsilon_{yy}) = \tau \cdot \frac{\Delta a}{a_0}$ $(\Delta a = a - a_0)$

⇨ $H = U - \tau a - \tau a_0$ Constant, neglect

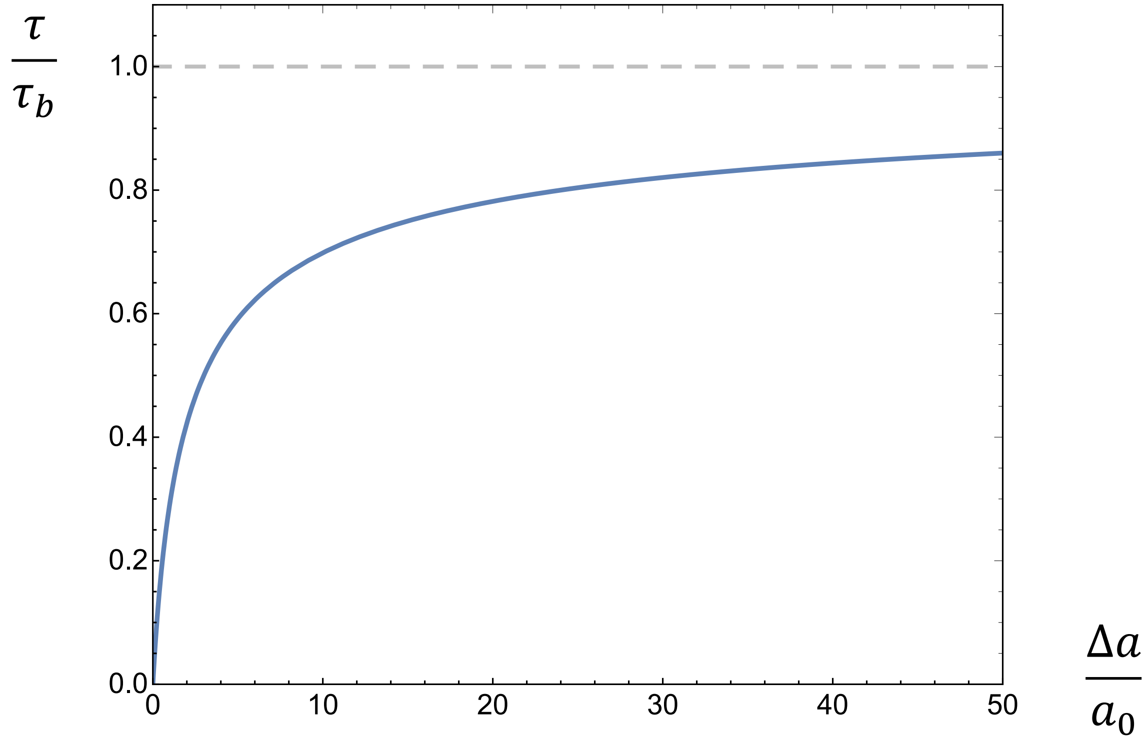
⇨ $H = \frac{3}{4} k_{sp} (l - l_0)^2 - \tau \frac{\sqrt{3}}{4} l^1$

Network under External Stress

Equilibrium: $dH = 0$ ⇨ $\frac{dH}{dl} = 0$

⇨ $$l = \frac{l_0}{1 - \tau/\tau_b}$$ $\tau_b = \sqrt{3}k_{sp}$

Catastropic blow-up ($l \to \infty$) as $\tau \to \tau_b$



Network under External Stress

What if $\tau < 0$ (still uniform)?

Deformation can be non-uniform ⇒ Huge number of variables

⇒ Need all $\frac{\partial H}{\partial l_i} = 0$

To solve this problem:

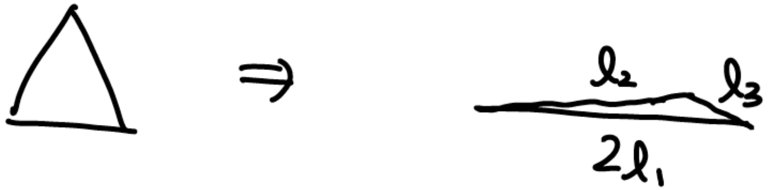
1. Numerical solution
2. Calculus of variation
3. Mode analysis

Network under External Stress

Mode 1: Uniform contraction (same as extension)

$$\Rightarrow \quad H_u^{eq}(\tau) = \frac{3}{4}k_{sp}l_0^2\left(1 - \frac{1}{1-\tau/\tau_b}\right) - \tau\frac{\sqrt{3}}{4}\frac{l_0^2}{(1-\tau/\tau_b)^2}$$

Mode 2: Plaquette collapse (zero area)



$$2l_1 = l_2 + l_3$$

Assume that all plaquettes collapse in the same way

$$H_c = U - \tau a = \frac{1}{2}k_{sp}[(2l_1 - l_0)^2 + (l_2 - l_0)^2 + (2l_1 - l_2 - l_0)^2]\cdot\frac{1}{2}$$

$$dH_c = 0 \quad \Leftrightarrow \quad \frac{\partial H_c}{\partial l_1} = 0 \qquad \frac{\partial H_c}{\partial l_2} = 0$$

Network under External Stress

$$\frac{\partial H_c}{\partial l_2} = 0 \quad \Rightarrow \quad l_1 = l_2$$

$$\frac{\partial H_c}{\partial l_1} = 0 \quad \Rightarrow \quad l_1 = \frac{2}{3} l_0$$

$$\Rightarrow \quad \boxed{H_c^{eq} = \frac{1}{12} k_{sp} l_0^2}$$

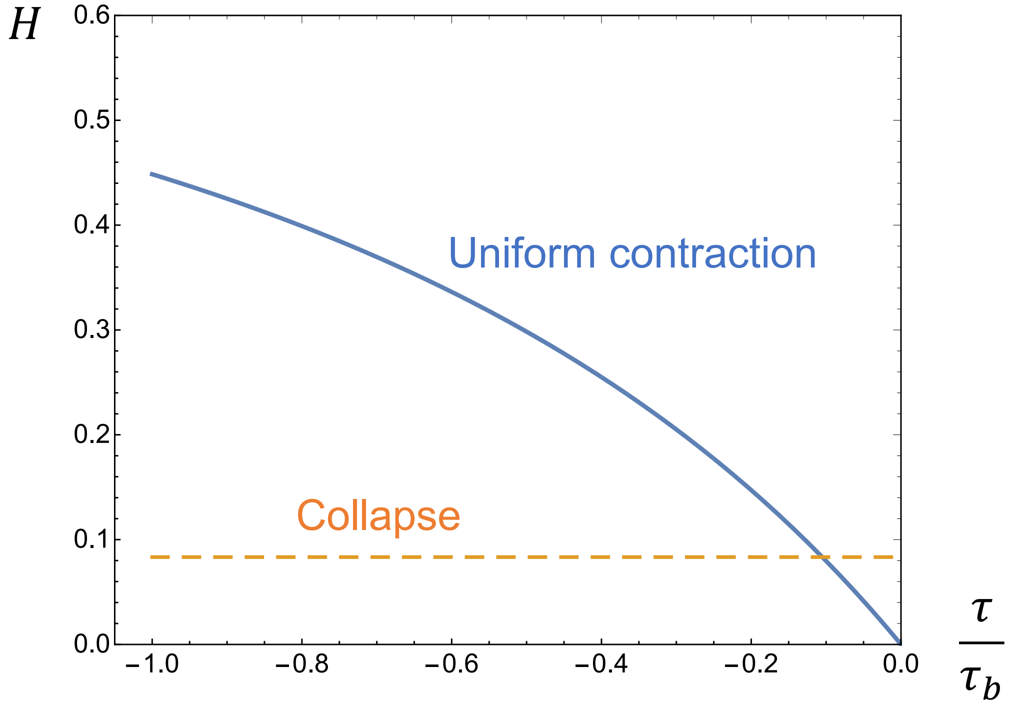
$$H_c^{eq} = H_u^{eq} \quad \Rightarrow \quad \tau_c = \frac{1}{8} \tau_b$$

$$l(\tau_c) = \frac{8}{9} l_0$$ Still uniform

$\Rightarrow$

$\tau_c < \tau < 0$ Uniform contraction

$\tau < \tau_c$ Collapse



Network under External Stress

Note:

1. Only 2 modes analyzed, others may have lower enthalpy and lead to instability first from uniform contraction
2. Network must deform from uniform contraction to collapse mode through intermediate states, these might have higher enthalpy (energy barrier)