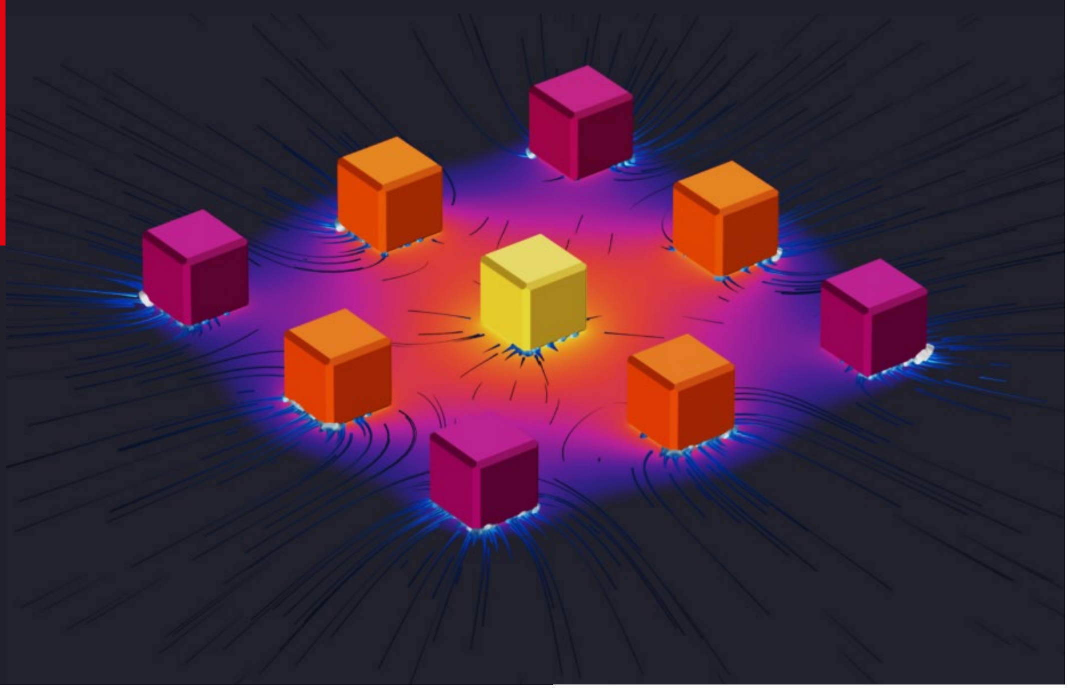
Nanoscale Heat Transfer (and Energy Conversion)

ME469

Instructor: Giulia Tagliabue



École
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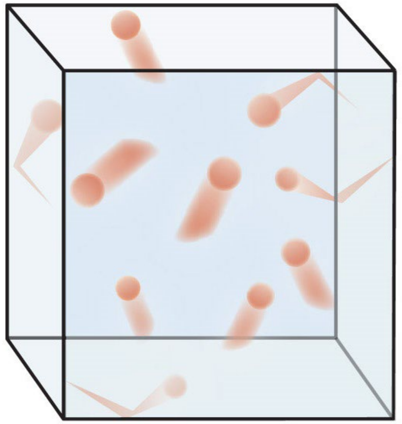
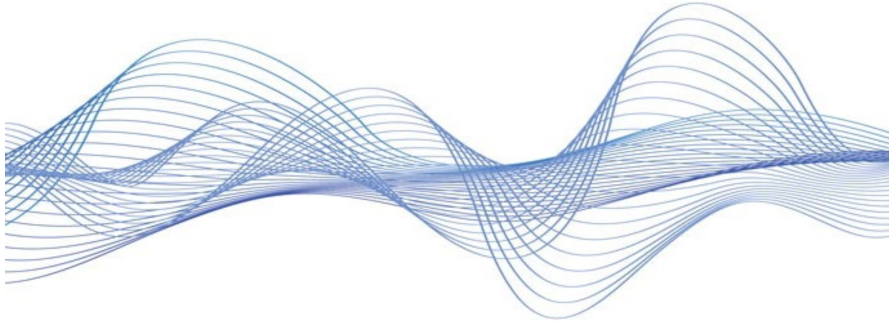
Spring Semester 2020

Wave-particle Duality

The wave nature of material particles gives rise to quantum mechanical effects!



Particle View



Wave View

Energy ⟵ $\boldsymbol{E = h\nu = \hbar\omega}$ ⟶ **Frequency, Amplitude**

Momentum ⟵ $\boldsymbol{p = \hbar k = \frac{h}{\lambda}}$ ⟶ **Wavevector**

$$h = 6.6 \cdot 10^{-34} Js$$
$$\hbar = h/2\pi$$

Describing the Material Waves – The Schrodinger Equation*

To understand the dynamics of the material waves (and hence of the associated particles) we use the Schrodinger equation:

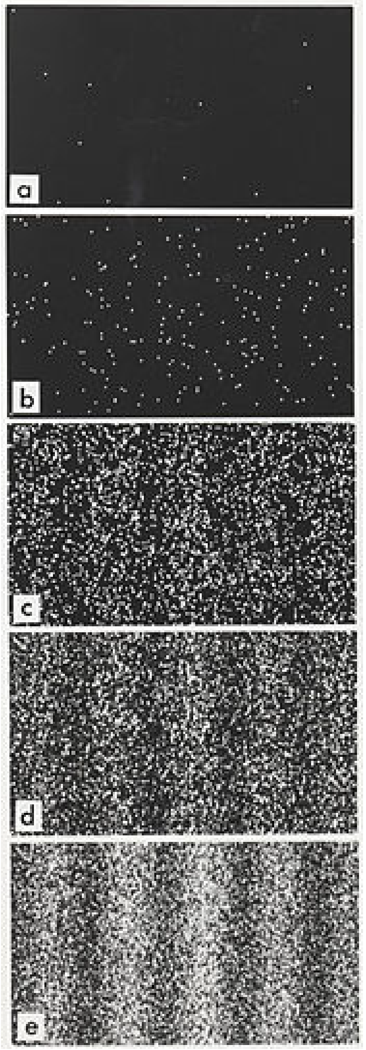
$$i\hbar\frac{\partial\Psi_t}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + U\right]\Psi_t$$

$U = potential\ energy\ landscape$

$\Psi_t(\vec{r}, t) = wavefunction$

$$\int_{-\infty}^{+\infty}\Psi_t{\Psi_t}^{*}dx = 1$$

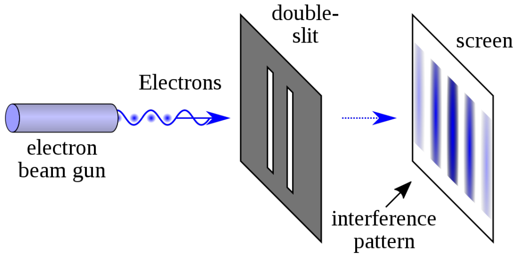
$\Psi_t{\Psi_t}^{*} = probability\ density\ function$



Steady-state Schrodinger equation (eigenvalue equation)

$$\left[-\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r}) + \boldsymbol{U}\Psi(\vec{r})\right] = E\Psi(\vec{r})$$

Together with the problem symmetry determines the energy levels and the wavefunctions



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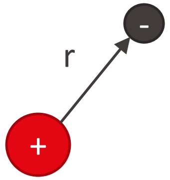
*Normally we would model the physics and derive the equations. Here we proceed in the opposite way.
We start from the equation and we show that the physics that follows corresponds to the observed physical phenomena

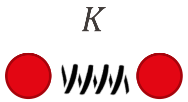
Summary of Solutions of Schrodinger Equation

	Potential	Wavefunction	Energy Levels	Quantum Numbers	Degeneracy
Free Particle	$\boldsymbol{U = 0}$	$\Psi_t = Ae^{-i(\omega t+kx)} + Be^{-i(\omega t-kx)}$ $k = \surd(2mE/\hbar\hat{}2\ )$	$E = \frac{\hbar^2}{2m}k^2$	-	-
1D Infinite Potential Well	**$U = 0$ for $0 <$ $x < L$ $U = \infty$ for x < 0 and x $> L$**	$\Psi_{n,t}(x,t) = \sqrt{\frac{2}{L}} sin\left(\frac{n\pi x}{L}\right) e^{-i\omega t}$	$E_n = \frac{1}{2m}\left(\frac{\hbar n\pi}{L}\right)^2$	n	none
2D Infinite Potential Well		$\Psi_{n_x,n_y,t}(x,y,t) = \frac{2}{D} sin\left(\frac{n_x\pi}{L_x}x\right) sin\left(\frac{n_y\pi}{L_y}y\right) e^{-i\omega t}$	$E_{n_x,n_y} = \frac{(\hbar\pi)^2}{2m}\left[\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2\right]$	n_x , n_y	Depends on $n_{i,j}$
Harmonic Oscillator (1D)	$\boldsymbol{U = \frac{Kx^2}{2}}$	$\Psi_{n,t}(x,t) = \left[\sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{2^n n!}\right]^{1/2} H_n\left(\left[\frac{m\omega x^2}{\hbar}\right]^{1/2}\right) e^{-\frac{m\omega x^2}{2\hbar}} e^{-i\omega t}$	$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$	n	none
Rigid Rotor		$Y_l^m(\theta,\varphi) = \varepsilon i^l \left[\frac{(2l+1)(l-\lvert m\rvert)!}{4\pi(l+\lvert m\rvert)!}\right]^{1/2} P_l^{\lvert m\rvert} cos\theta e^{im\varphi}$	$E_l = \frac{\hbar^2}{2I}l(l+1)$	l, m	$2l + 1$
Hydrogen Atom (3D)	$U = -\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}$	$\Psi_{n,l,m}(r,\theta,\varphi) = R_{n,l}(r)Y_l^m(\theta,\varphi)$	$E_n = -\frac{Z^2e^4\mu}{8\epsilon_0^2n^2h^2}$	n, l, m	Depends on n

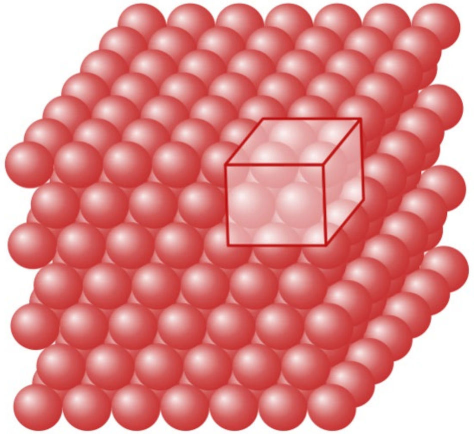
From Individual Elements to Periodic Arrangements



Individual Atom
(electron)

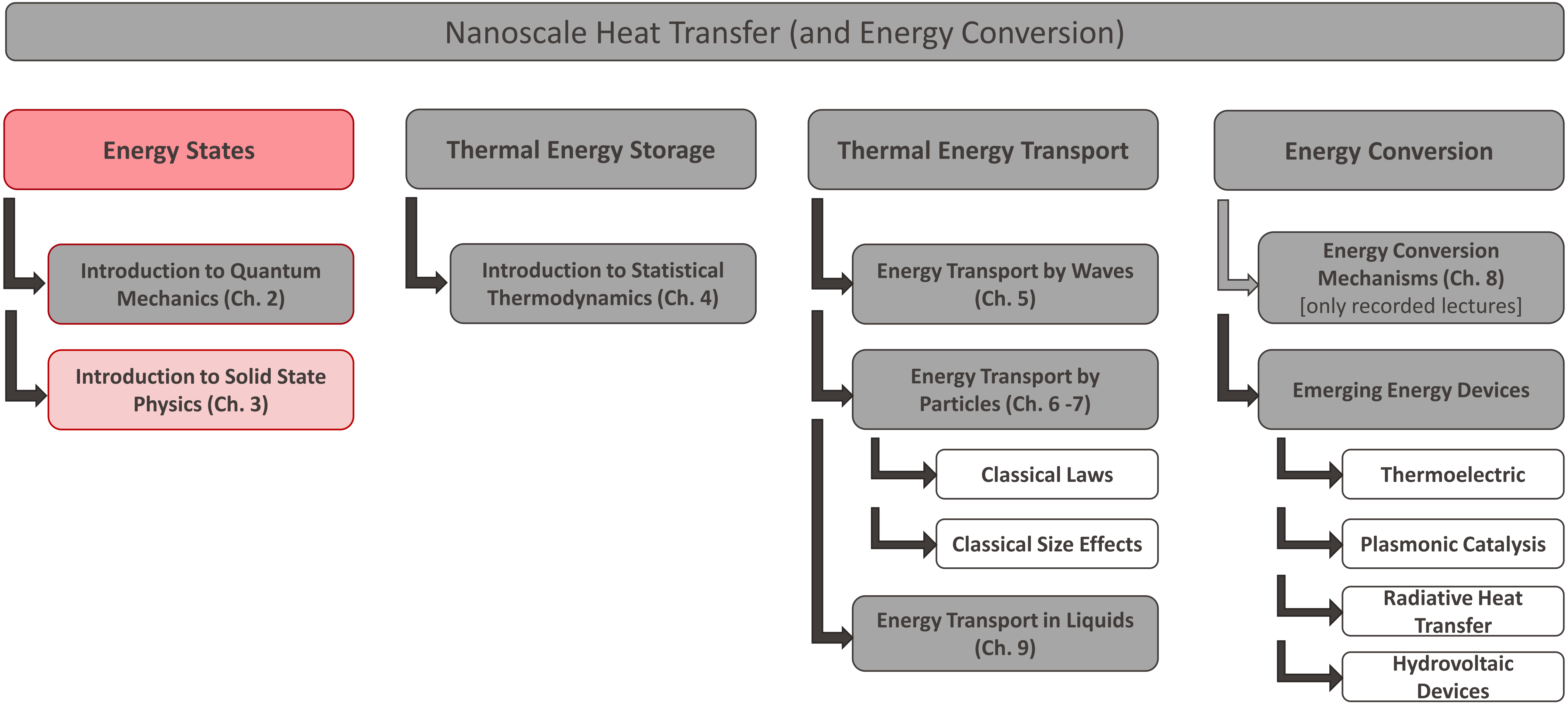


Individual Molecule
Single pair of Particles
(phonons)



3D Periodic Arrangement

This Lecture



Nanoscale Heat Transfer (and Energy Conversion)

Energy States
- Introduction to Quantum Mechanics (Ch. 2)
- Introduction to Solid State Physics (Ch. 3)

Thermal Energy Storage
- Introduction to Statistical Thermodynamics (Ch. 4)

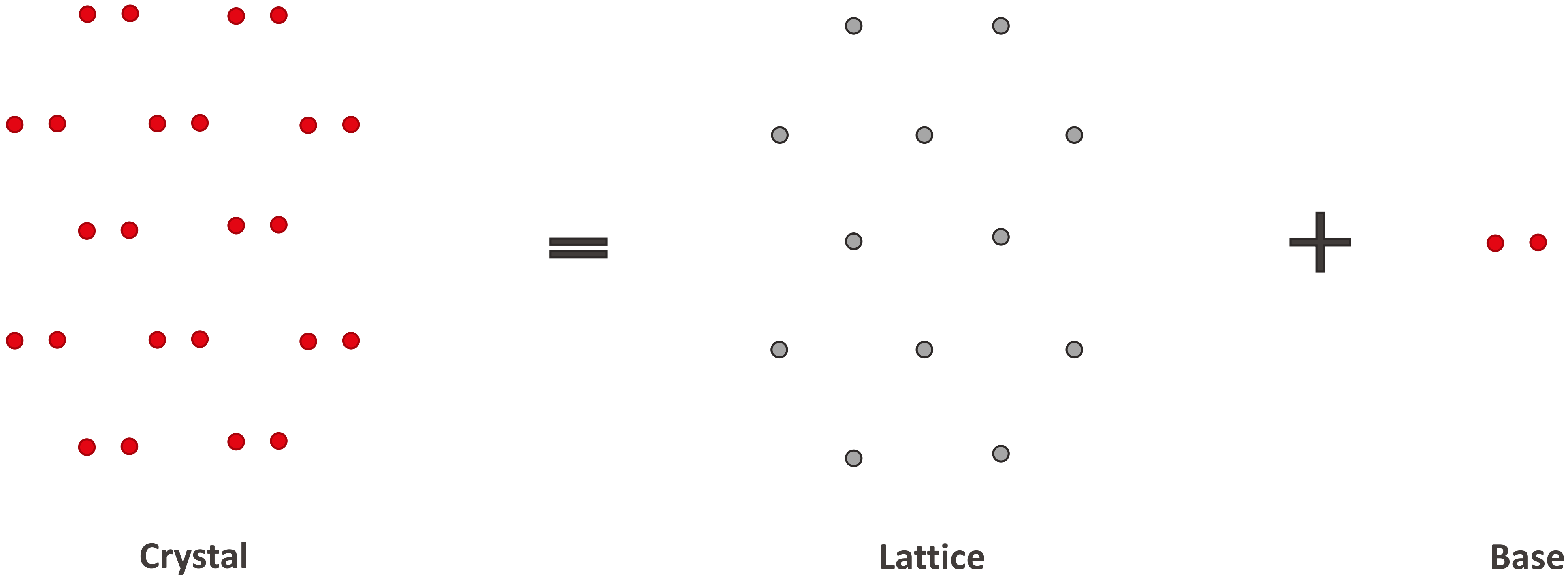
Thermal Energy Transport
- Energy Transport by Waves (Ch. 5)
- Energy Transport by Particles (Ch. 6 -7)
  - Classical Laws
  - Classical Size Effects
- Energy Transport in Liquids (Ch. 9)

Energy Conversion
- Energy Conversion Mechanisms (Ch. 8) [only recorded lectures]
- Emerging Energy Devices
  - Thermoelectric
  - Plasmonic Catalysis
  - Radiative Heat Transfer
  - Hydrovoltaic Devices

In This Lecture…

- **Crystal Structure, the Real and Reciprocal Lattice, Miller Indices**
- Electron Energy States in Periodic Lattices – Intro to Periodic Potential

Crystal Structure and the Real Lattice

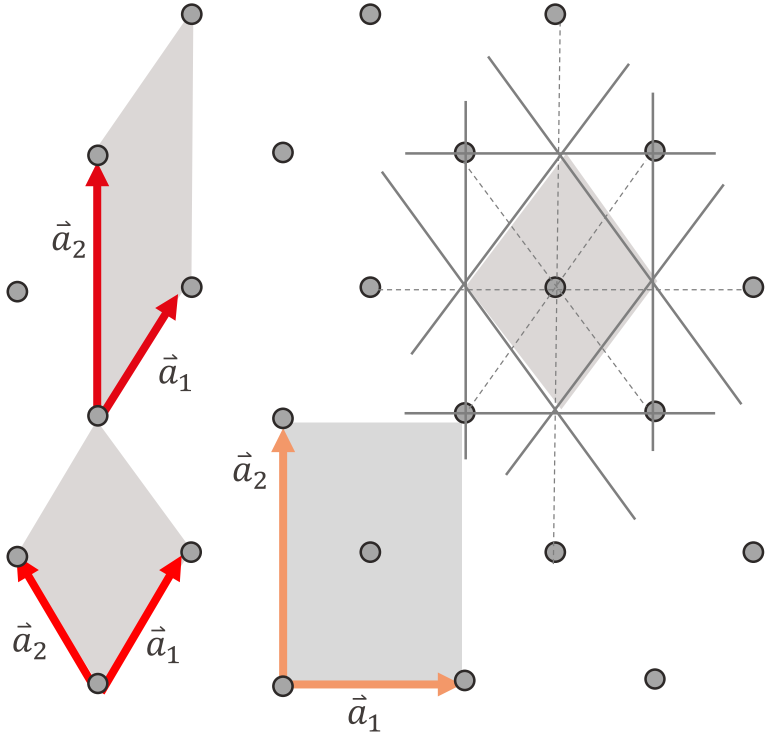


A crystal is a 3D periodic arrangement of atoms.
To describe it we use a **lattice**, i.e. a periodic array of points that replicates the periodicity.
Importantly, the periodicity of the crystal structure might become apparent only if we consider the regular repetition of a group of more the one atom.
This is called the **base**.

➡ **Crystal = lattice + base**

Crystal Structure and the Real Lattice

Lattice



- A **Bravais lattice** is an infinite array of discrete points with an **arrangement AND orientation** that appears EXACTLY the same from whichever of the points the array is viewed
- A 3D Bravais lattice consists of all points with position vectors $\vec{R}$ of the form:

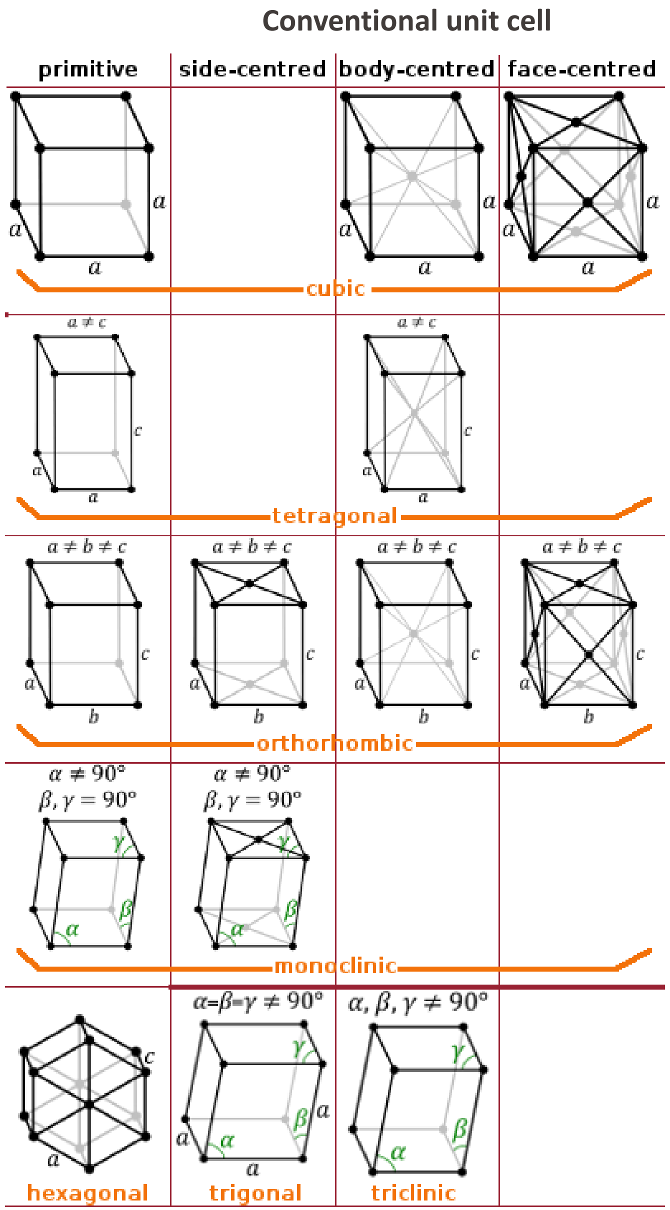
$$\vec{R} = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3$$

We observe that the choice of the primitive lattice vectors (vectors connecting nearest neighbors in the lattice) entails a degree of arbitrariness.
We define a **primitive unit cell** as a parallelepiped defined by three primitive lattice vectors that contains only one lattice point.

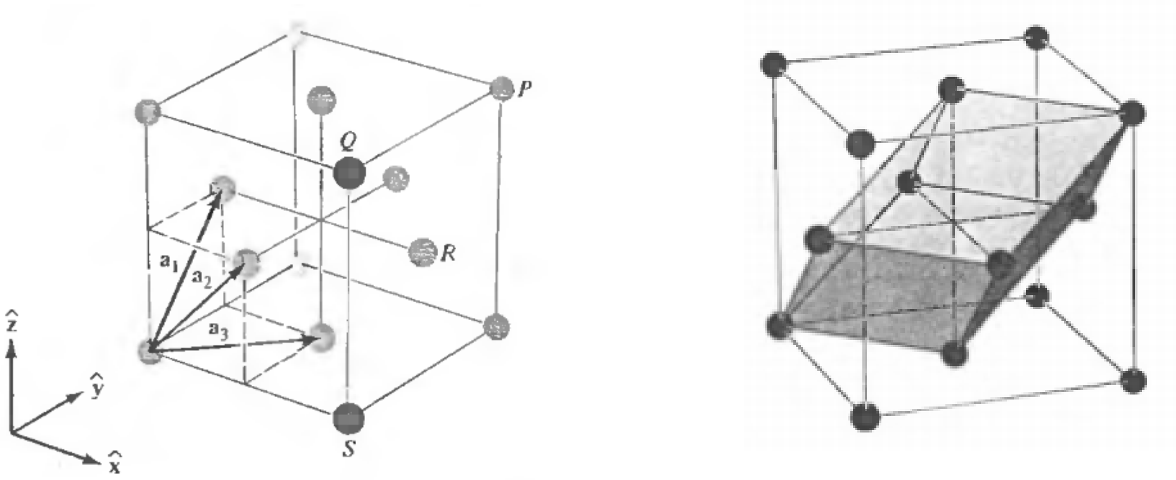
A construction that always ensures we have identified a primitive unit cell is the Wigner-Seitz construction. It consists in connecting all neighboring points surrounding an arbitrary lattice point and then to draw the bisecting plane perpendicular to each connection line.

Sometimes we describe the lattice by a conventional unit cell, that can contain more than one lattice point but has a more intuitive shape. It is important to recognize this is not a primitive unit cell.

Crystal Structure and the Real Lattice



- There are **14 types of Bravais lattice** with varying degrees of symmetry (side lengths and angles)
- The fundamental requirements that the primitive cell can fill the ENTIRE space limits the rotational symmetries to $2\pi, \pi, 2\pi/3, \pi/2$ and $\pi/3$
- For each lattice we can identify the primitive vectors and the Wigner-Seitz Primitive cell.
  - E.g. for the face centered cubic (FCC) lattice

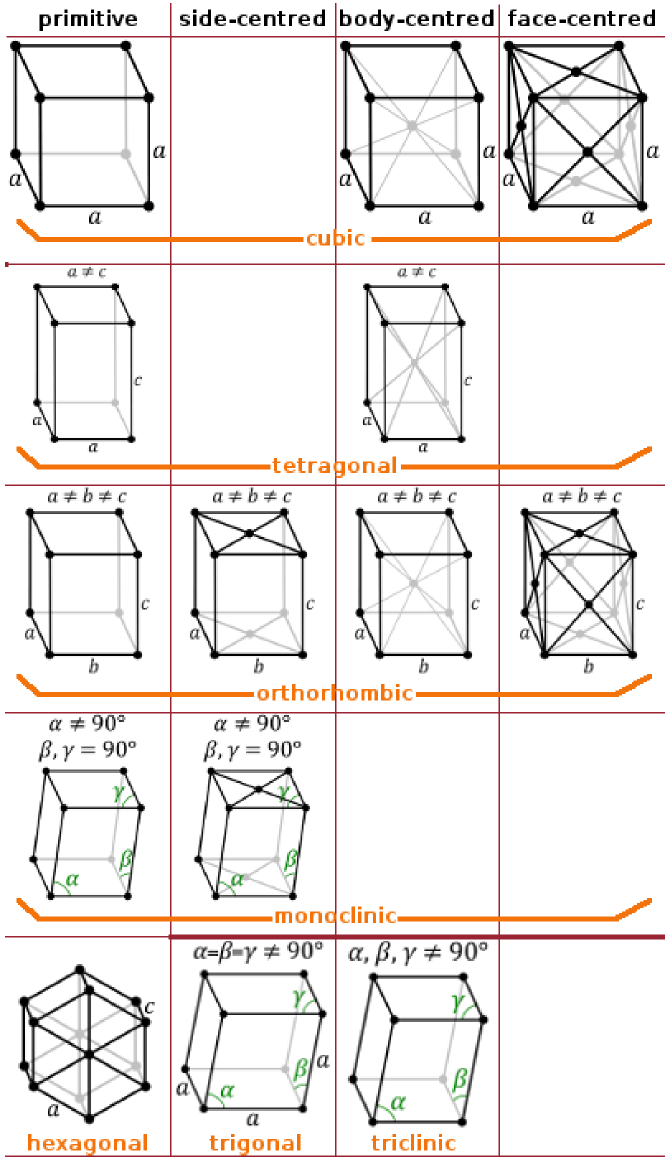


$$\mathbf{a}_1 = \frac{a}{2}(\hat{\mathbf{y}} + \hat{\mathbf{z}}), \quad \mathbf{a}_2 = \frac{a}{2}(\hat{\mathbf{z}} + \hat{\mathbf{x}}), \quad \mathbf{a}_3 = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}).$$

- Elements such as Ag, Au, Al, Cu crystallize with FCC structure

Crystal Structure and the Real Lattice

Conventional unit cell



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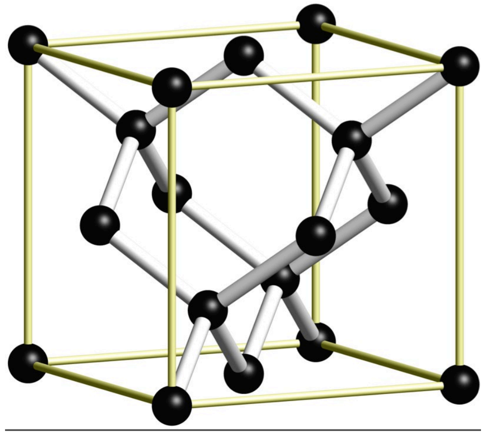


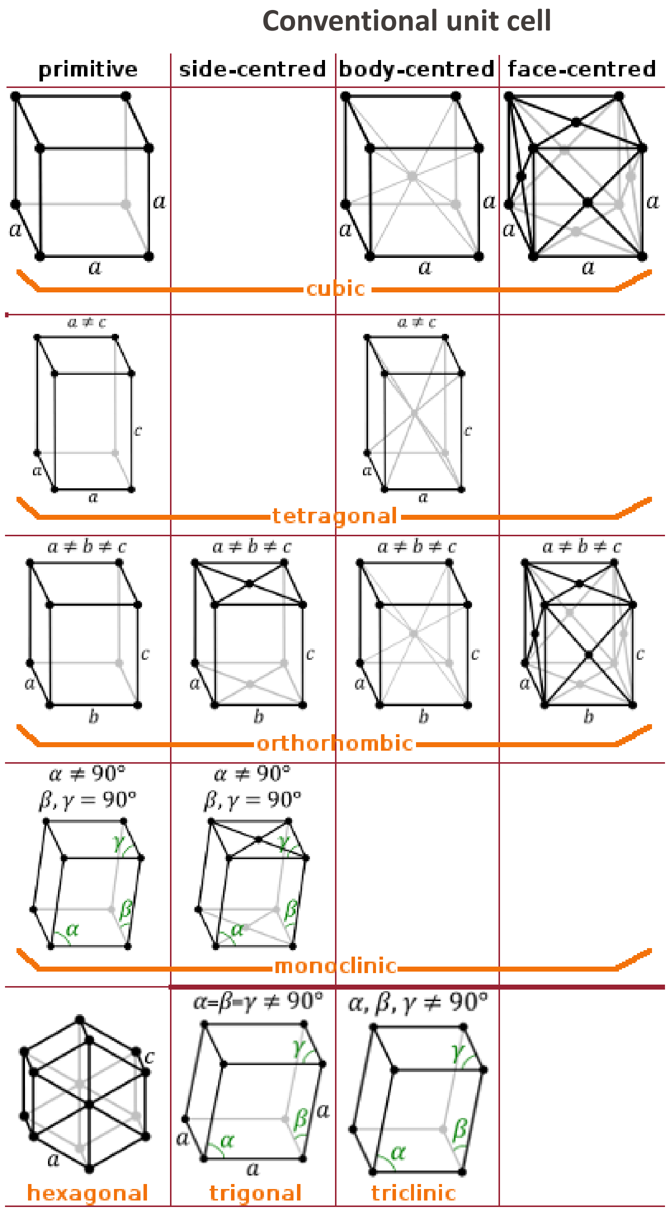
$$\mathbf{a}_1 = \frac{a}{2}(\hat{\mathbf{y}} + \hat{\mathbf{z}}), \quad \mathbf{a}_2 = \frac{a}{2}(\hat{\mathbf{z}} + \hat{\mathbf{x}}), \quad \mathbf{a}_3 = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}).$$

Base atoms : 0 and $a/4(\hat{x} + \hat{y} + \hat{z})$

➡ *Zincblende structure*

- Elements such as Ag, Au, Al, Cu, Pd, Pt, Rh crystallize with FCC structure
- A diamond structure is an FCC with a 2-atom base whose coordinates
- Elements such as C, Si, Ge crystallize with diamond structure
- A Zincblende structure is like a diamond structure but the two atom forming the base are different
- Elements such as GaAs, GaP, SiC, CdS, crystallize with Zincblende structure

Crystal Structure and the Real Lattice



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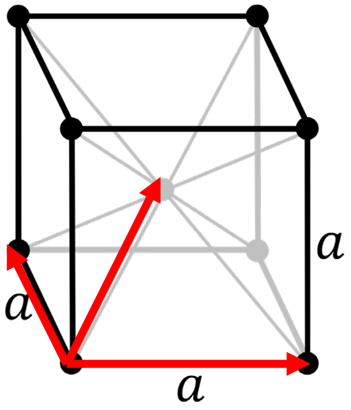
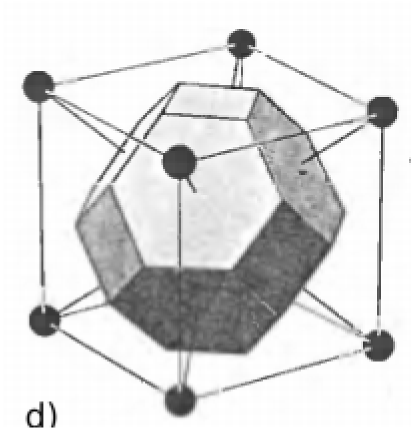
- The fundamental requirements that the primitive cell can fill the ENTIRE space limits the rotational symmetries to $2\pi, \pi, 2\pi/3, \pi/2$ and $\pi/3$
- For each lattice we can identify the primitive vectors and the Wigner-Seitz Primitive cell.
  - E.g. for the **body centered cubic (BCC)** lattice

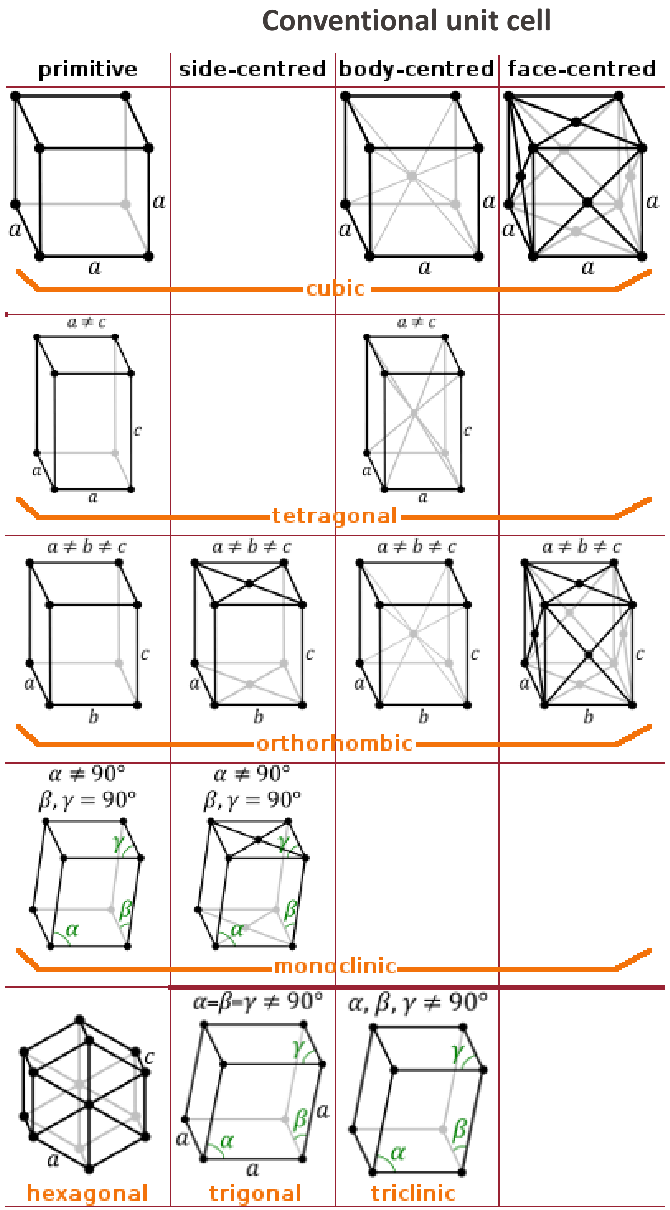




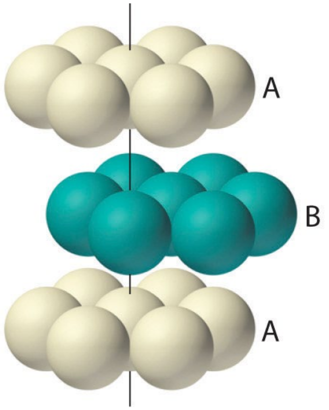
$$\vec{a}_1 = a\,\hat{x} \qquad \vec{a}_2 = a\,\hat{y} \qquad \vec{a}_3 = \frac{a}{2}(\hat{x}+\hat{y}+\hat{z})$$

- Elements such as Cr, Fe, K, Li, Mo, Na, W crystallize with body centered cubic (BCC)

Crystal Structure and the Real Lattice

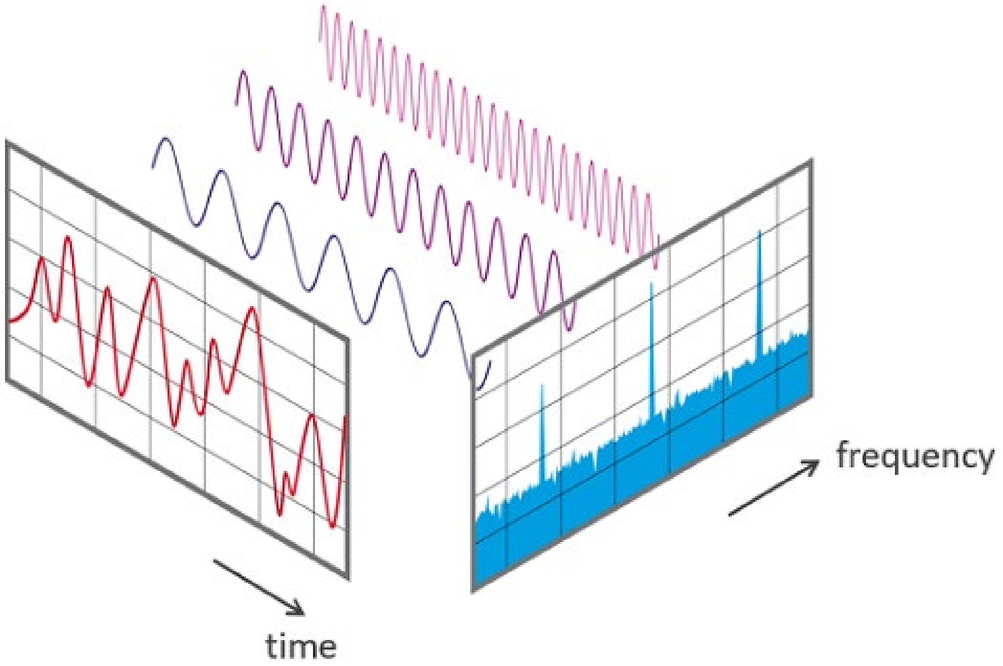


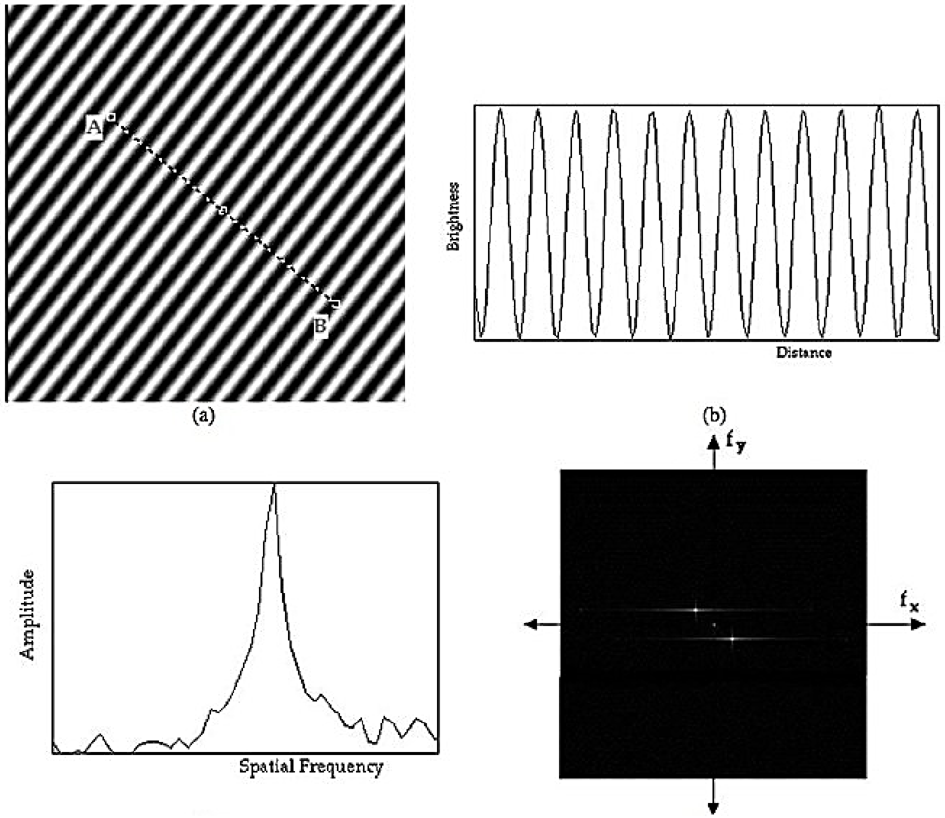
- There are **14 types of Bravais lattice** with varying degrees of symmetry (side lengths and angles)
- The fundamental requirements that the primitive cell can fill the ENTIRE space limits the rotational symmetries to $2\pi, \pi, 2\pi/3, \pi/2$ and $\pi/3$
- Another important lattice structure is the hexagonal closed-pack structure. This is NOT a Bravais lattice but can be obtained as two interpenetrating simple hexagonal Bravais lattices.



- Elements such as Mg, Hf, Ru, Ti, Zn crystallize with hexagonal closed-packed structure

Fourier Transforms and the Reciprocal Lattice





- It is often convenient to calculate the Fourier transform of a signal (temporal or spatial) to identify the frequencies contained in it.
- Considering that a lattice is a periodic structure, by calculating its Fourier transform we can identify the fundamental frequencies that compose it

Fourier Transforms and the Reciprocal Lattice

We remember that a periodic function $f(x)$ with spatial period a can be expressed as:

$$f(x) = \sum_{n=-\infty}^{\infty} \left(a_n sin\frac{2\pi n}{a}x + b_n cos\frac{2\pi n}{a}x\right) = \sum_{n=-\infty}^{\infty} \left(a'_n e^{ink_x} + b'_n e^{-ink_x}\right) = \sum_{n=-\infty}^{\infty} \left(a'_n - ib'_n\right) e^{ink_x} \qquad \text{where} \qquad k_x = \frac{2\pi}{a}$$

If we now consider a 3D function that is invariant with any translational lattice vector $\vec{R} = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3$

$$f(\vec{r}) = \sum_{\vec{G}} f_G e^{i\vec{r}\cdot\vec{G}} \qquad \text{where} \qquad \vec{G} = m_1\vec{b}_1 + m_2\vec{b}_2 + m_3\vec{b}_3 \qquad f_G = \frac{1}{V}\int_{unit\ cell} f(\vec{r})e^{-i\vec{r}\cdot\vec{G}}$$

Because of the periodicity: $f(\vec{r}) = \sum_{\vec{G}} f_G e^{i\vec{r}\cdot\vec{G}} = \sum_{\vec{G}} f_G e^{i(\vec{r}+\vec{R})\cdot\vec{G}} = f(\vec{r}+\vec{R})$ ➡ $e^{i\vec{R}\cdot\vec{G}} = 1$ ➡ $\overrightarrow{a_i}\cdot\overrightarrow{b_j} = 2\pi\delta_{ij}$

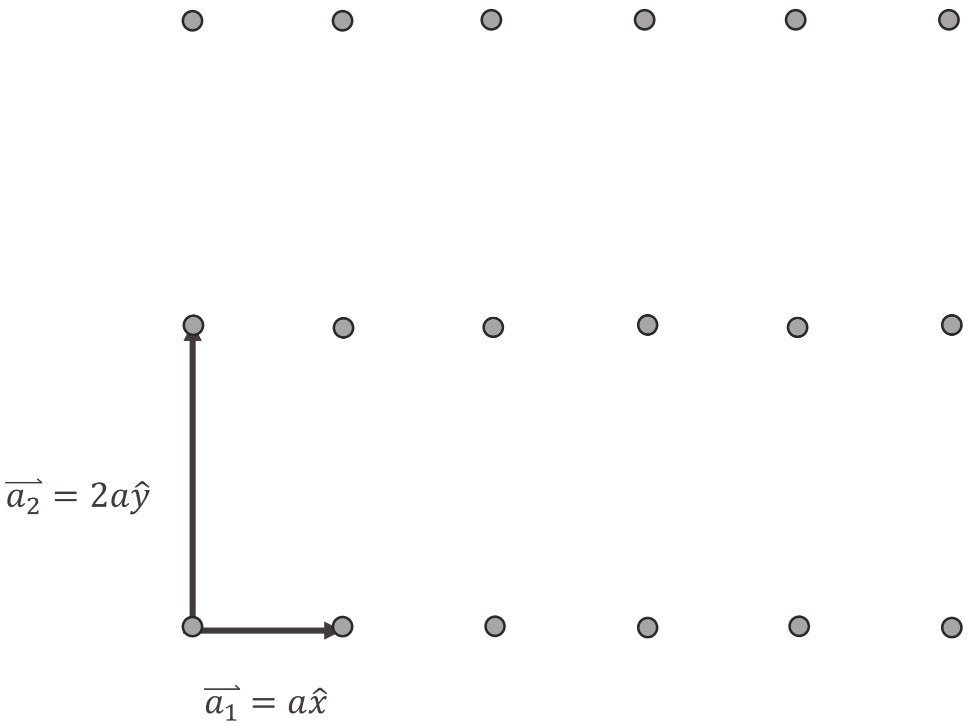
Thus we have:
$\overrightarrow{a_2}\cdot\overrightarrow{b_1} = 0$
$\overrightarrow{a_3}\cdot\overrightarrow{b_1} = 0$
➡ $\vec{b}_1 = c\,\vec{a}_2\times\vec{a}_3$

In addition: $\overrightarrow{a_1}\cdot\overrightarrow{b_1} = 2\pi$ ➡ $c = \frac{2\pi}{\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)}$

We can thus define:

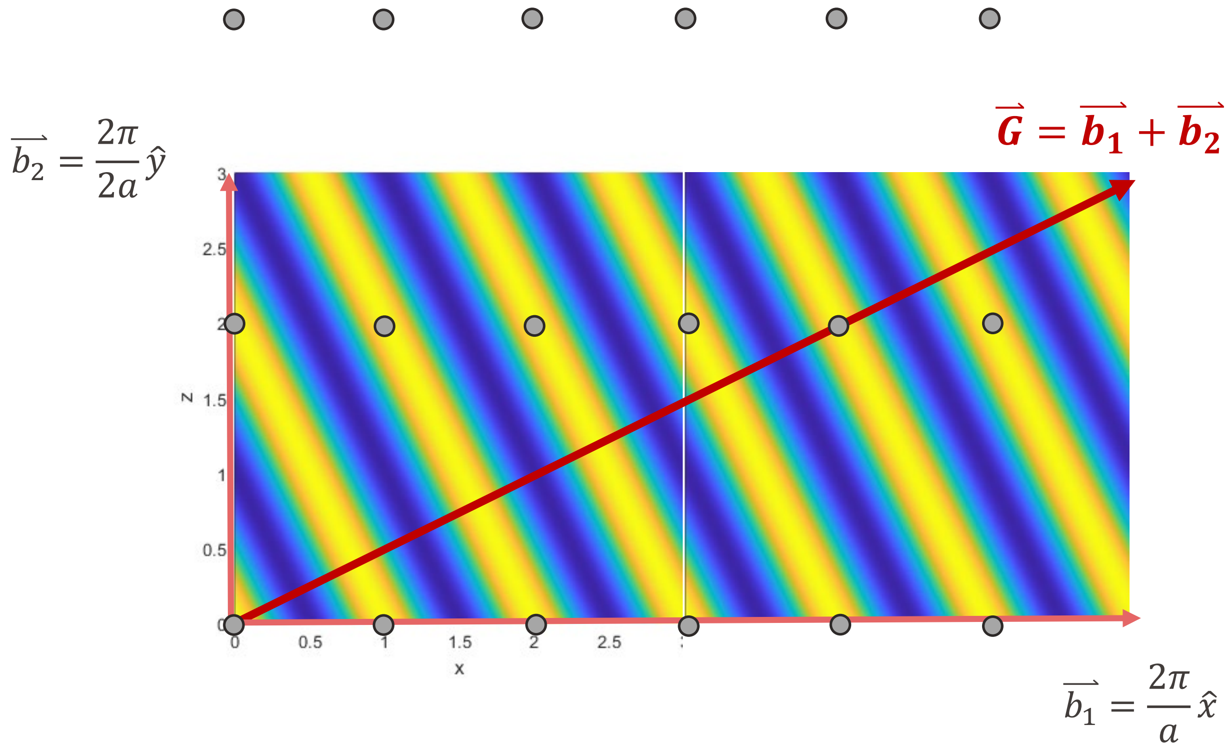
$$\vec{b}_1 = 2\pi\frac{\vec{a}_2\times\vec{a}_3}{\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)} \qquad \vec{b}_2 = 2\pi\frac{\vec{a}_3\times\vec{a}_1}{\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)} \qquad \vec{b}_3 = 2\pi\frac{\vec{a}_1\times\vec{a}_2}{\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)} \qquad \text{where} \qquad V = \vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)$$

Fourier Transforms and the Reciprocal Lattice



$\overrightarrow{a_2} = 2a\hat{y}$

$\overrightarrow{a_1} = a\hat{x}$



$\overrightarrow{b_2} = \frac{2\pi}{2a}\hat{y}$

$\boldsymbol{\overrightarrow{G} = \overrightarrow{b_1} + \overrightarrow{b_2}}$

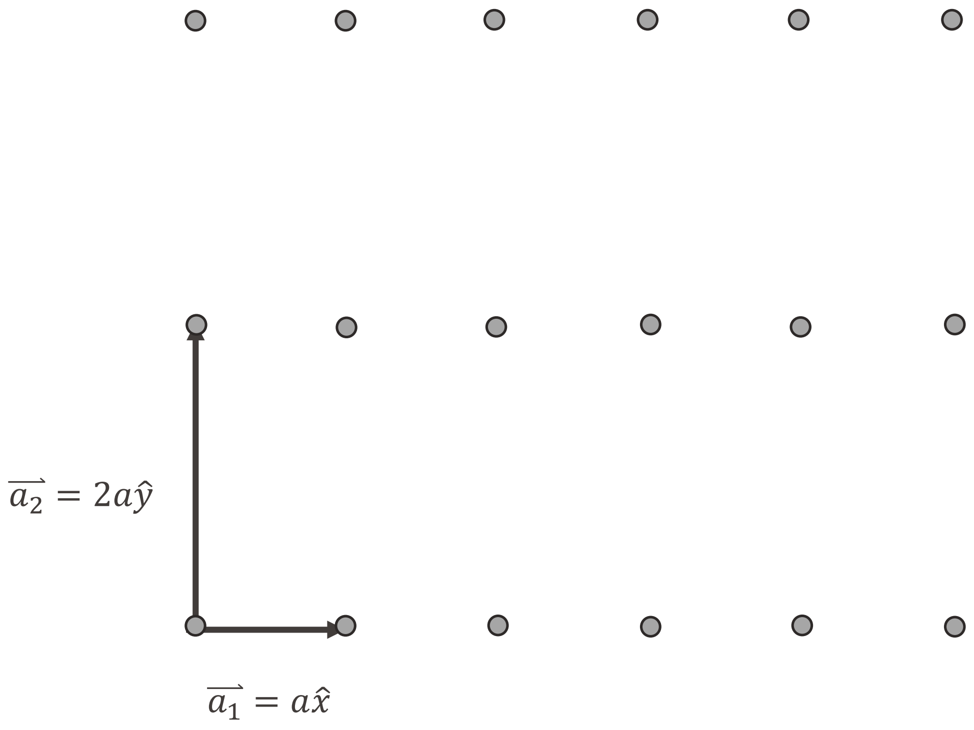
$\overrightarrow{b_1} = \frac{2\pi}{a}\hat{x}$

In 2D we simply have :

$$\overrightarrow{a_1} \cdot \overrightarrow{b_1} = 2\pi \quad \Rightarrow \quad \overrightarrow{b_1} = \frac{2\pi}{a}\hat{x}$$

$$\overrightarrow{a_2} \cdot \overrightarrow{b_2} = 2\pi \quad \Rightarrow \quad \overrightarrow{b_2} = \frac{2\pi}{2a}\hat{y}$$

Fourier Transforms and the Reciprocal Lattice



$\overrightarrow{a_2} = 2a\hat{y}$

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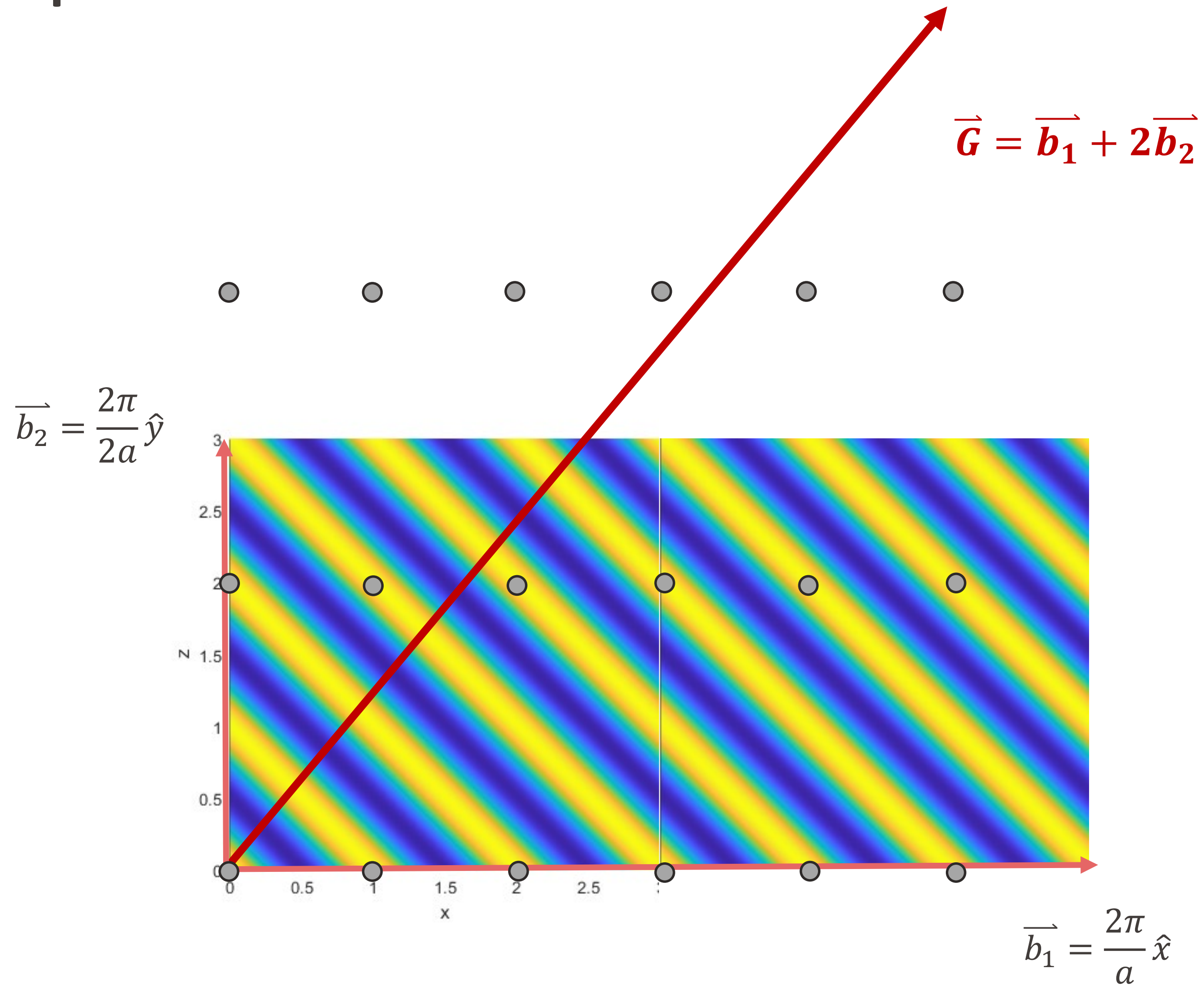

$\overrightarrow{G} = \overrightarrow{b_1} + 2\overrightarrow{b_2}$

$\overrightarrow{b_2} = \frac{2\pi}{2a}\hat{y}$

$\overrightarrow{b_1} = \frac{2\pi}{a}\hat{x}$

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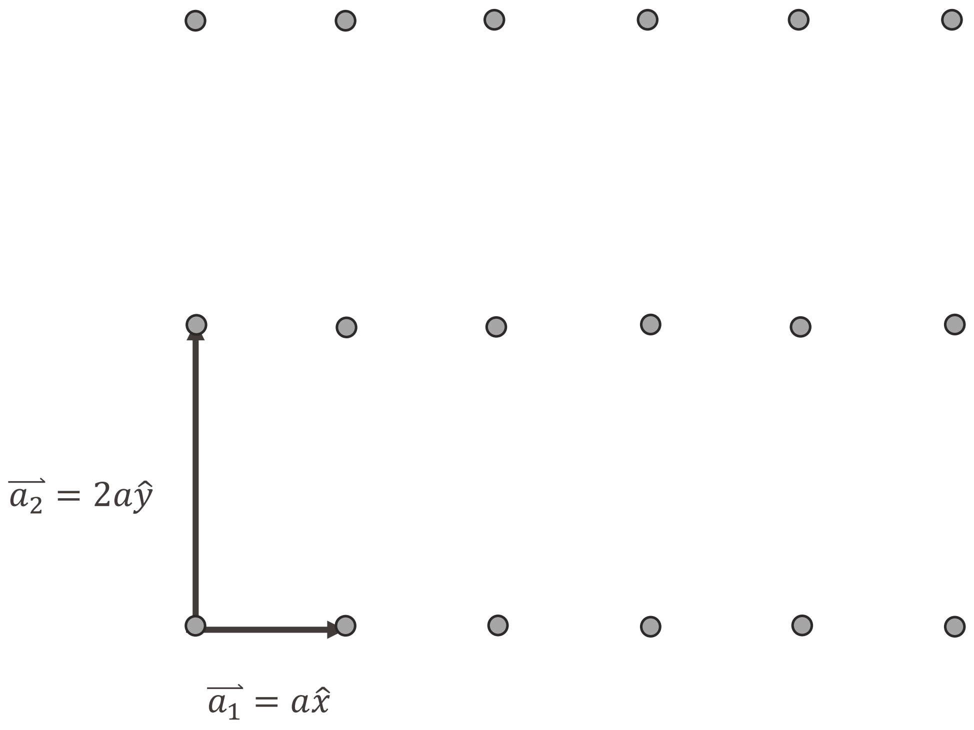
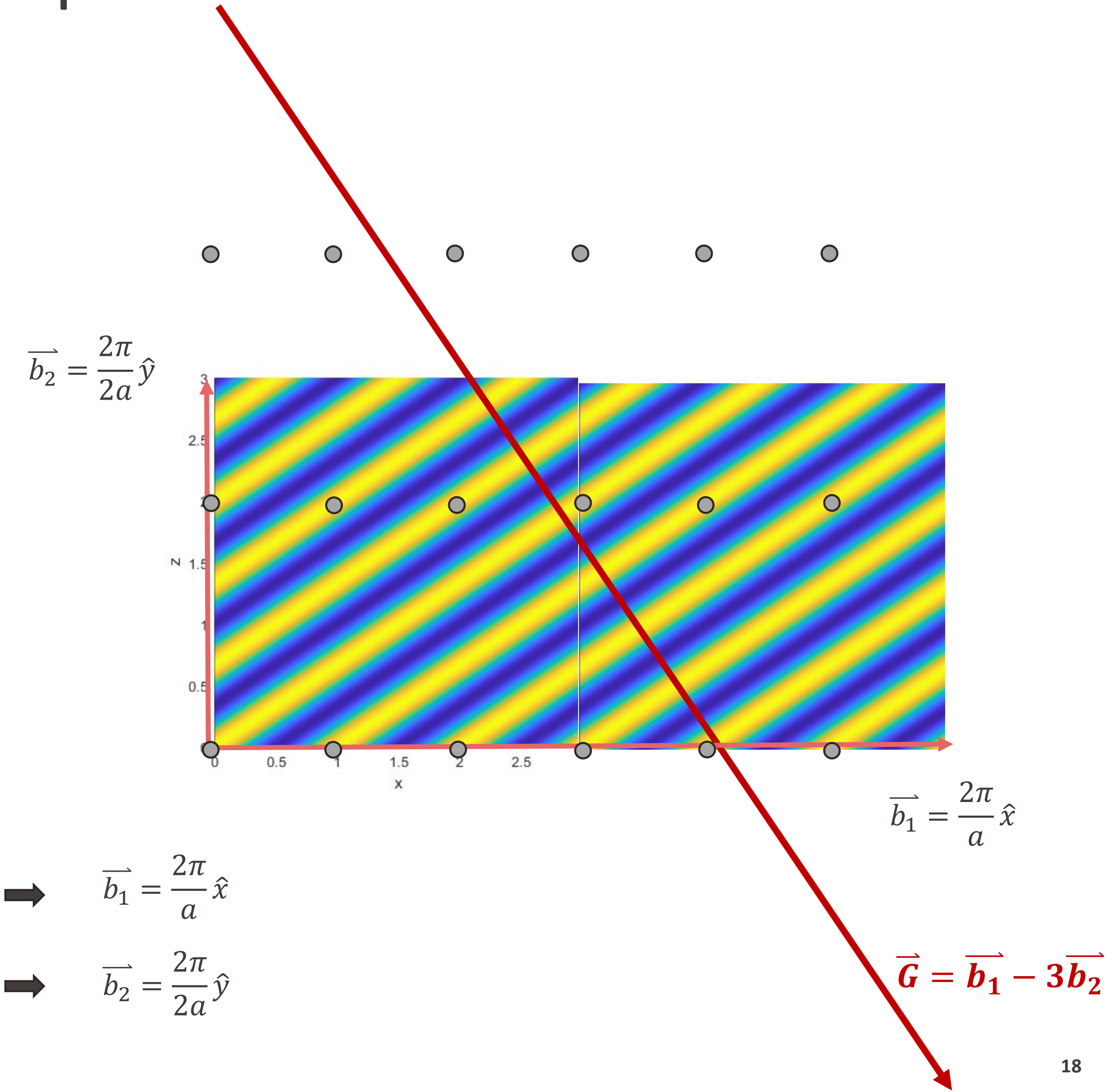
$$\overrightarrow{a_2} \cdot \overrightarrow{b_2} = 2\pi \quad \Rightarrow \quad \overrightarrow{b_2} = \frac{2\pi}{2a}\hat{y}$$

Fourier Transforms and the Reciprocal Lattice



$\overrightarrow{a_2} = 2a\hat{y}$

$\overrightarrow{a_1} = a\hat{x}$



$\overrightarrow{b_2} = \frac{2\pi}{2a}\hat{y}$

$\overrightarrow{b_1} = \frac{2\pi}{a}\hat{x}$

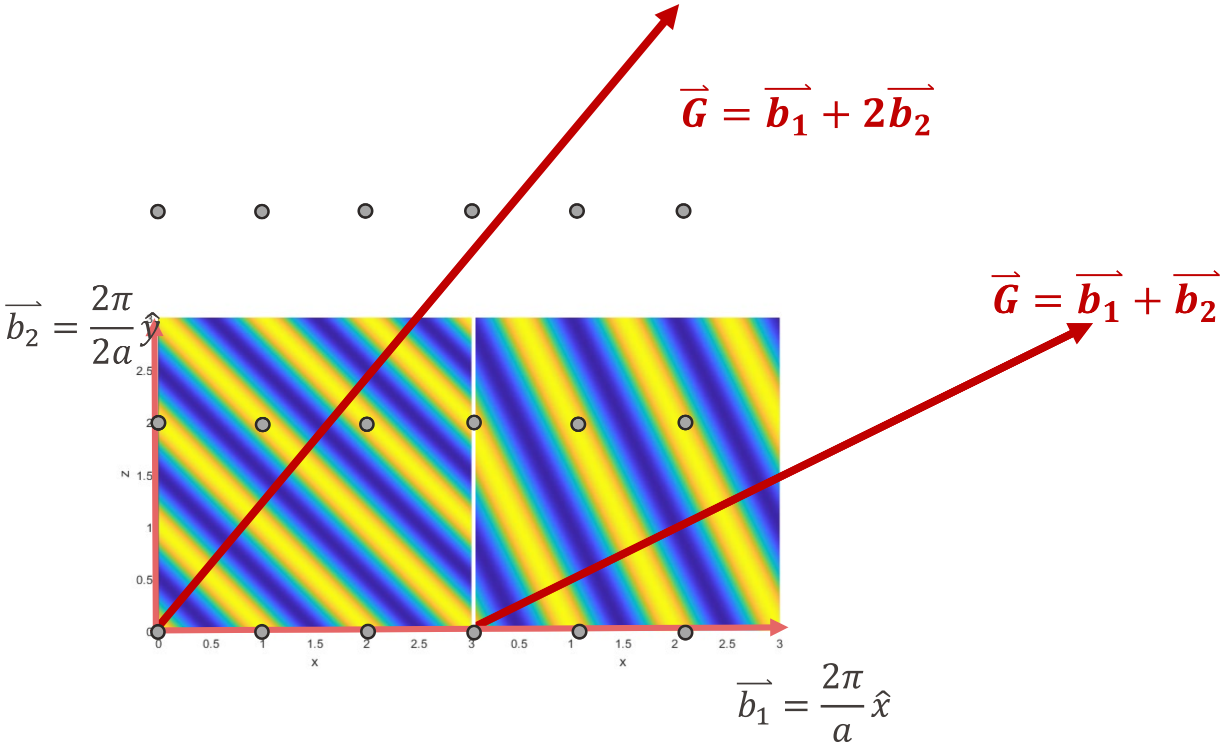
$\boldsymbol{\vec{G} = \vec{b_1} - 3\vec{b_2}}$

In 2D we simply have :

$\overrightarrow{a_1} \cdot \overrightarrow{b_1} = 2\pi \Rightarrow \overrightarrow{b_1} = \frac{2\pi}{a}\hat{x}$

$\overrightarrow{a_2} \cdot \overrightarrow{b_2} = 2\pi \Rightarrow \overrightarrow{b_2} = \frac{2\pi}{2a}\hat{y}$

Fourier Transforms and the Reciprocal Lattice



$$\vec{G} = \overrightarrow{b_1} + 2\overrightarrow{b_2}$$

$$\vec{G} = \overrightarrow{b_1} + \overrightarrow{b_2}$$

$$\overrightarrow{b_2} = \frac{2\pi}{2a}\hat{y}$$

$$\overrightarrow{b_1} = \frac{2\pi}{a}\hat{x}$$

Reciprocal lattice vectors identify waves with periodicity equal to that of the lattice along different spatial directions.

The spacing between lattice points along different spatial directions is different.

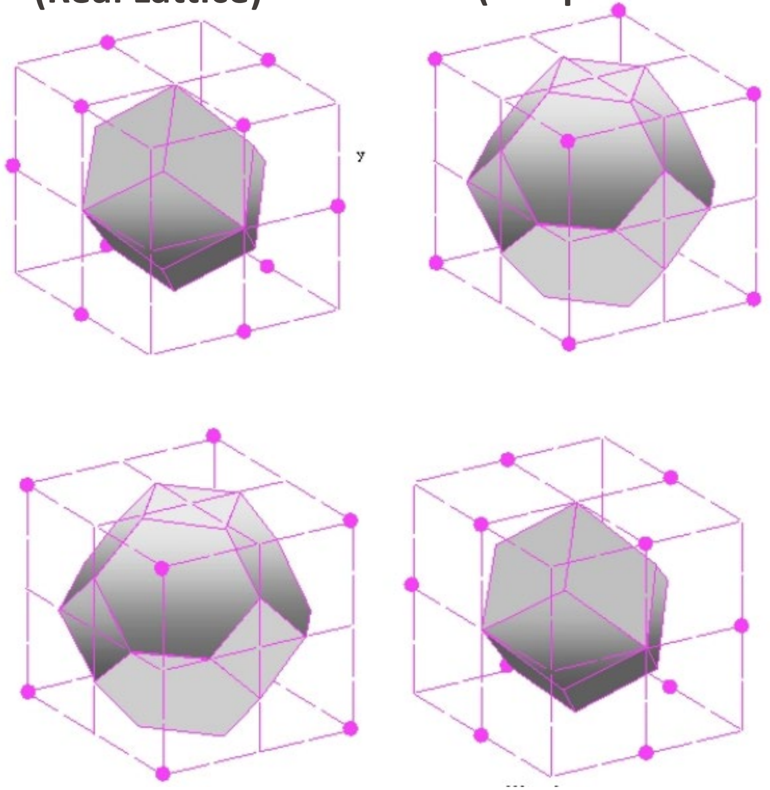
Reciprocal Lattice and Brillouin Zone

Reciprocal lattice = the set of all wavevectors $\vec{G}$ that yield plane waves with the periodicity of a given Bravais lattice.

Like a real lattice the reciprocal lattice has a set of primitive vectors, $\vec{b}_1, \vec{b}_2, \vec{b}_3$ and it is possible to define its Wigner-Seitz cell, which is called **Brillouin zone**.

Wigner-Seitz Cell (Real Lattice)

Brillouin Zone (Reciprocal Lattice)



FCC

BCC

BCC

FCC

$$\vec{R} = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3$$

$$\vec{G} = m_1\vec{b}_1 + m_2\vec{b}_2 + m_3\vec{b}_3$$

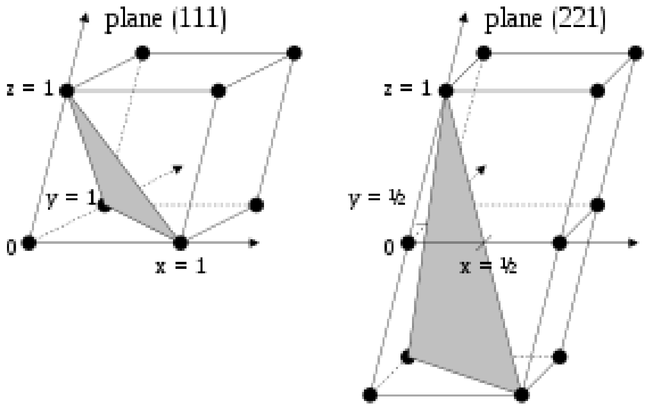
$$\vec{b}_1 = 2\pi\frac{\vec{a}_2 \times \vec{a}_3}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$$

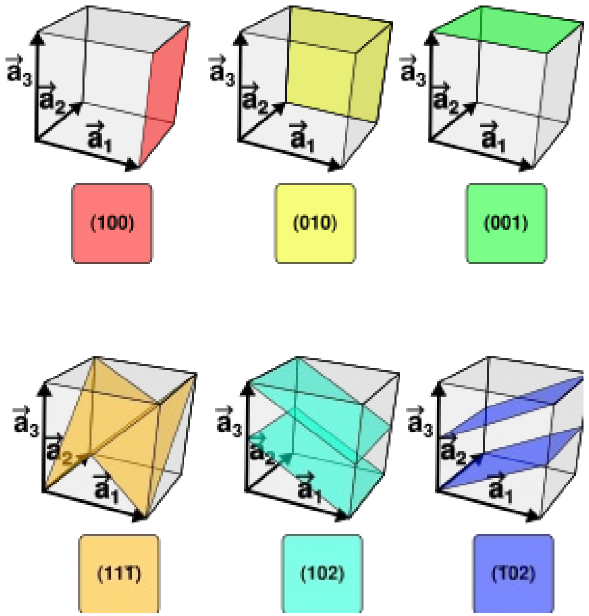
$$\vec{b}_2 = 2\pi\frac{\vec{a}_3 \times \vec{a}_1}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$$

$$\vec{b}_3 = 2\pi\frac{\vec{a}_1 \times \vec{a}_2}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$$

Note: the wavevectors lying on the surface of the Brillouin zone define the Nyquist condition for the 3D crystal. This means that any wavevector larger than the Brillouin zone can be expressed with a wavevector inside the Brillouin zone.

Miller Indices





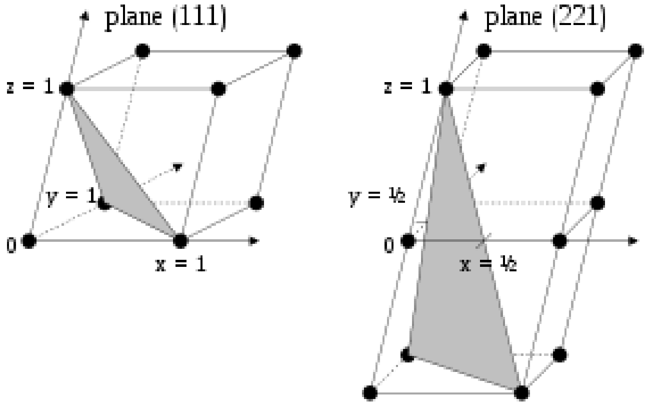
For any family of lattice planes separated by a distance d, there are reciprocal lattice vectors perpendicular to the planes, the shortest of which have a length of $2\pi/d$.

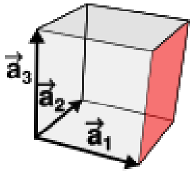
The *Miller Indices* of a lattice plane are the coordinates of the shortest reciprocal lattice vector normal to that plane, with respect to a specified set of reciprocal lattice vectors Thus **a Miller plane with indices $\boldsymbol{h}, \boldsymbol{l}, \boldsymbol{k}$ is normal to the reciprocal lattice vector $\vec{\boldsymbol{G}} = \boldsymbol{h}\vec{\boldsymbol{b}}_1 + \boldsymbol{l}\vec{\boldsymbol{b}}_2 + \boldsymbol{k}\vec{\boldsymbol{b}}_3$** (see slide 17)

To obtain the Miller indices of a plane (see image on the top right):

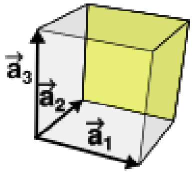
- Consider the real lattice and identify the intercepts of the desired plane with the axes formed by the primitive vectors $\vec{a}_i$
  - ➢ E.g. $1\vec{a}_1, 1\vec{a}_2, 2\vec{a}_3$ (or $1/2\vec{a}_1, 1/2\vec{a}_2, 1\vec{a}_3$)
- Take the reciprocal of the intercepts and reduce them to the three smallest integers that have the same ratio as the original set
  - ➢ E.g. $\left(\frac{1}{1}, \frac{1}{1}, \frac{1}{2}\right) \rightarrow \left(\frac{2}{2}, \frac{2}{2}, \frac{1}{2}\right) \rightarrow (2,2,1)$

Miller Indices

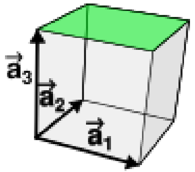




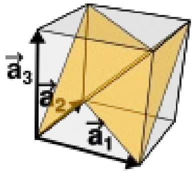
(100)



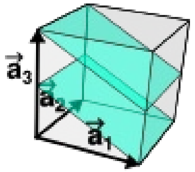
(010)



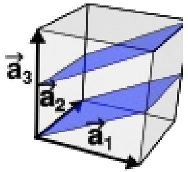
(001)



$(11\bar{1})$



(102)



$(\bar{1}02)$

We use the following notations:

- Miller plane: $(\boldsymbol{hlk})$
- If one of the plane indices is negative we write $(-\boldsymbol{h}, \boldsymbol{l}, \boldsymbol{k}) = (\bar{\boldsymbol{h}}\boldsymbol{lk})$
- Based on the real lattice symmetries, planes along different directions can exhibit the same atom arrangement. We call these **equivalent planes** and indicate them as $\{\boldsymbol{hkl}\}$
  - E.g. in a cubic lattice the planes (100), (010), (001), $(\bar{1}00)$, $(0\bar{1}0)$, $(00\bar{1})$
- Direction perpendicular to a Miller plane: $[\boldsymbol{hkl}]$
- All equivalent directions in a crystal are indicated as $\langle\boldsymbol{hkl}\rangle$

In This Lecture…

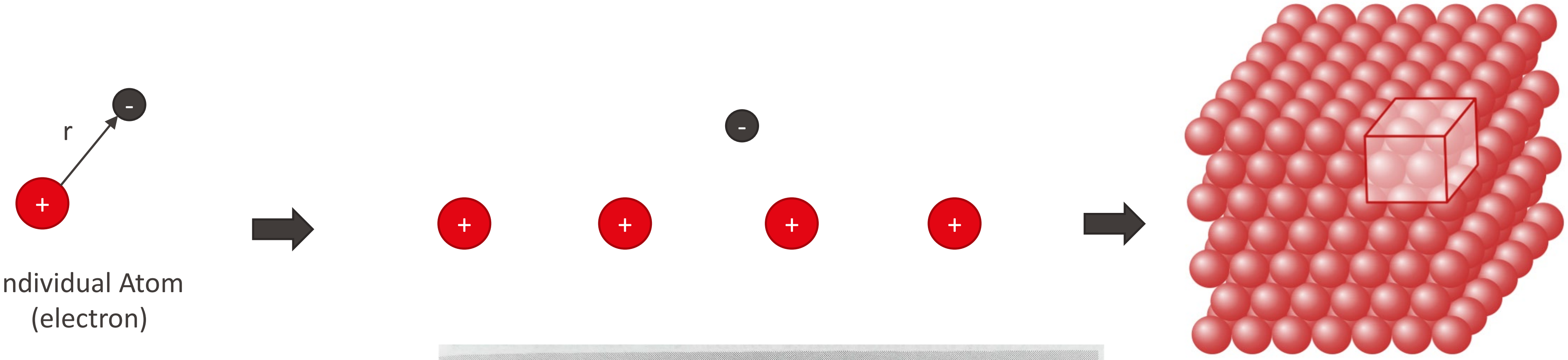
- Crystal Structure, the Real and Reciprocal Lattice, Miller Indices
- **Electron Energy States in Periodic Lattices – Intro to Periodic Potential**

Summary of Solutions of Schrodinger Equation (electrons)

	Potential	Wavefunction	Energy Levels	Quantum Numbers	Degeneracy
Free Particle	$U = 0$	$\Psi_t = Ae^{-i(\omega t+kx)} + Be^{-i(\omega t-kx)}$ $k = \sqrt{(2mE/\hbar^2)}$	$E = \frac{\hbar^2}{2m}k^2$	-	-
1D Infinite Potential Well	$U = 0$ for $0 < x < L$ $U = \infty$ for $x < 0$ and $x > L$	$\Psi_{n,t}(x,t) = \sqrt{\frac{2}{L}}\,sin\left(\frac{n\pi x}{L}\right)e^{-i\omega t}$	$E_n = \frac{1}{2m}\left(\frac{\hbar n\pi}{L}\right)^2$	n	none
2D Infinite Potential Well		$\Psi_{n_x,n_y,t}(x,y,t) = \frac{2}{D}\,sin\left(\frac{n_x\pi}{L_x}x\right)sin\left(\frac{n_y\pi}{L_y}y\right)e^{-i\omega t}$	$E_{n_x,n_y} = \frac{(\hbar\pi)^2}{2m}\left[\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2\right]$	n_x , n_y	Depends on $n_{i,j}$
Harmonic Oscillator (1D)	$U = \frac{Kx^2}{2}$	$\Psi_{n,t}(x,t) = \left[\sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{2^n n!}\right]^{1/2} H_n\left(\left[\frac{m\omega x^2}{\hbar}\right]^{1/2}\right)e^{-\frac{m\omega x^2}{2\hbar}}e^{-i\omega t}$	$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$	n	none
Rigid Rotor		$Y_l^m(\theta,\varphi) = \varepsilon i^l\left[\frac{(2l+1)(l-\lvert m\rvert)!}{4\pi(l+\lvert m\rvert)!}\right]^{1/2} P_l^{\lvert m\rvert} cos\theta e^{im\varphi}$	$E_l = \frac{\hbar^2}{2I}l(l+1)$	l, m	$2l + 1$
Hydrogen Atom (3D)	$U = -\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}$	$\Psi_{n,l,m}(r,\theta,\varphi) = R_{n,l}(r)Y_l^m(\theta,\varphi)$	$E_n = -\frac{Z^2e^4\mu}{8\epsilon_0^2 n^2 h^2}$	n, l, m	Depends on n

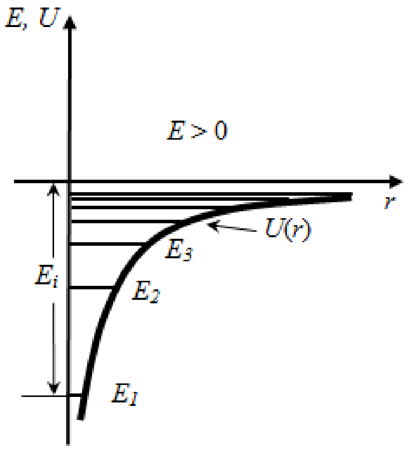
➡ The potential defines the wavefunctions and the energy states. In a crystal we will have a periodic potential.

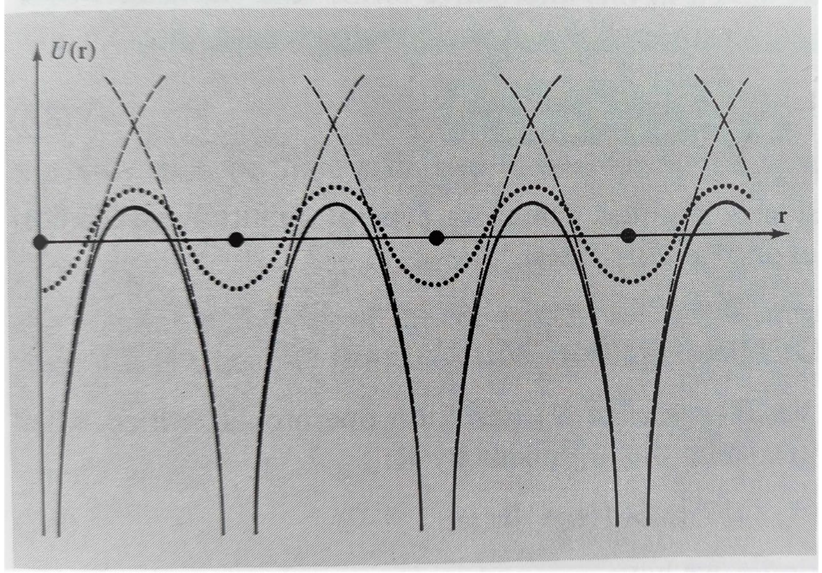
From Individuals to Crystals



Individual Atom
(electron)

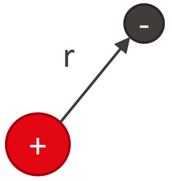
3D Periodic
Arrangement



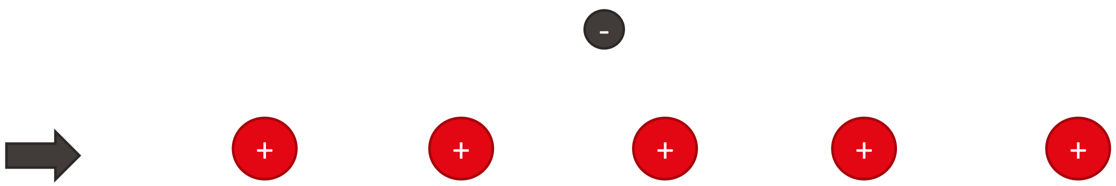


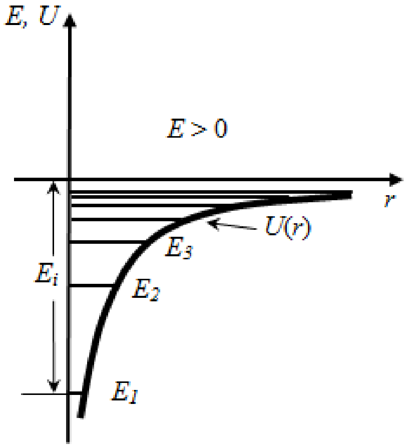
➡ **What are the electronic energy levels and wavefunctions in a periodic potential?**

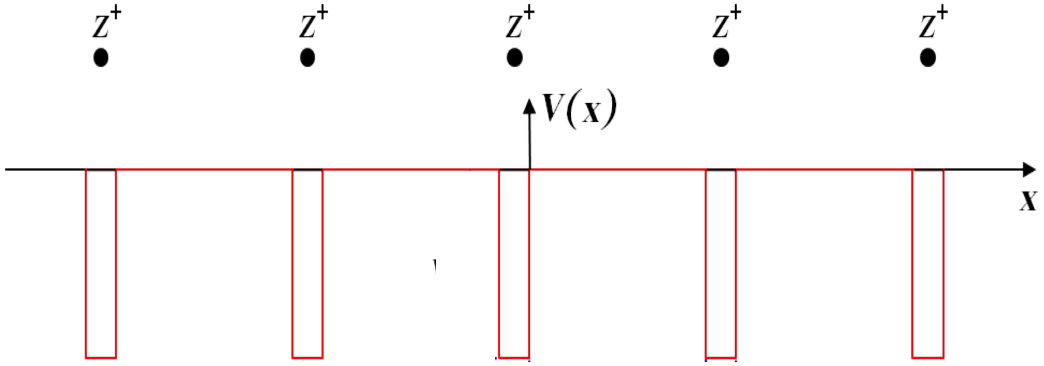
From Individuals to Crystals



Individual Atom
(electron)



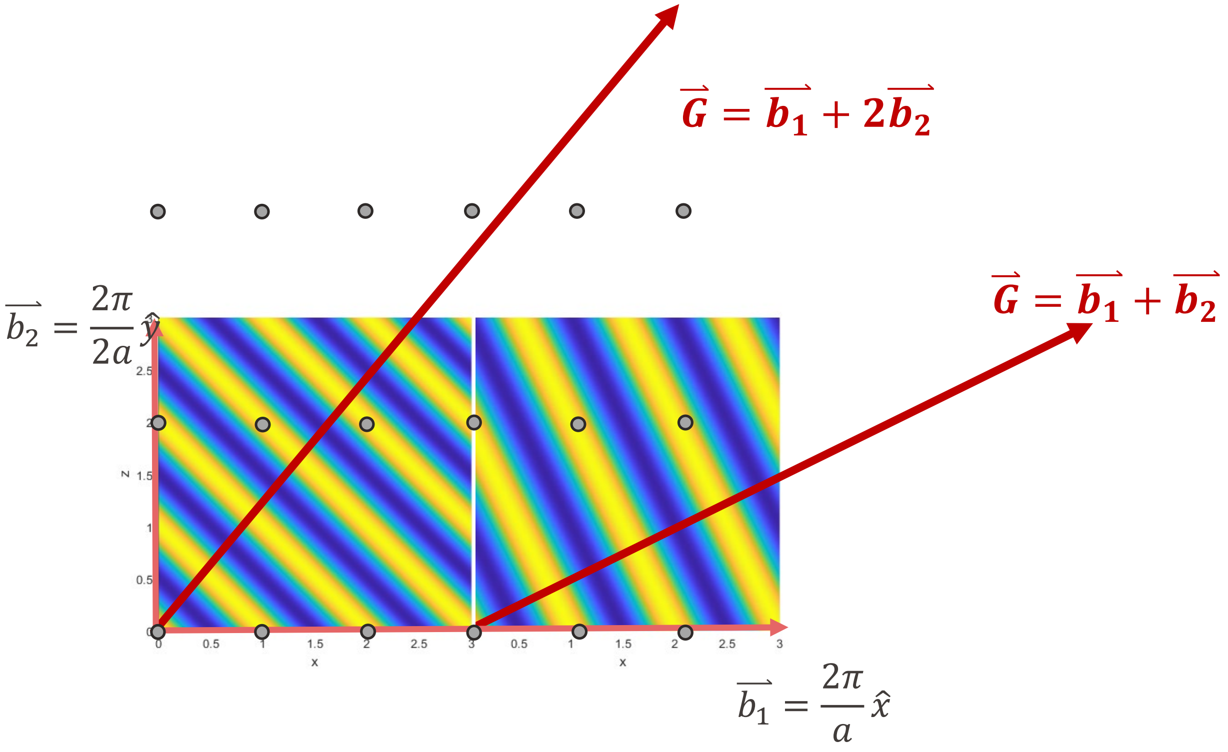




Kronig-Penney Potential

➡ **What are the electronic energy levels and wavefunctions in a periodic potential?**

From Individuals to Crystals



$\vec{G} = \vec{b_1} + 2\vec{b_2}$

$\vec{G} = \vec{b_1} + \vec{b_2}$

$\vec{b_2} = \frac{2\pi}{2a}\hat{y}$

$\vec{b_1} = \frac{2\pi}{a}\hat{x}$

Reciprocal lattice vectors identify waves with periodicity equal to that of the lattice along different spatial directions.

The spacing between lattice points along different spatial directions is different.

➡ **Different directions in the lattice have different periodic potentials!**

In the Next Lecture…

- **Electron Energy States in Periodic Lattices**
  - ➢ Band Structure and Material Properties
  - ➢ Density of States