Turbulence Through the Lens of Dynamical Systems and Chaos Theory

Omid Ashtari
Tobias M. Schneider

ME467—Turbulence
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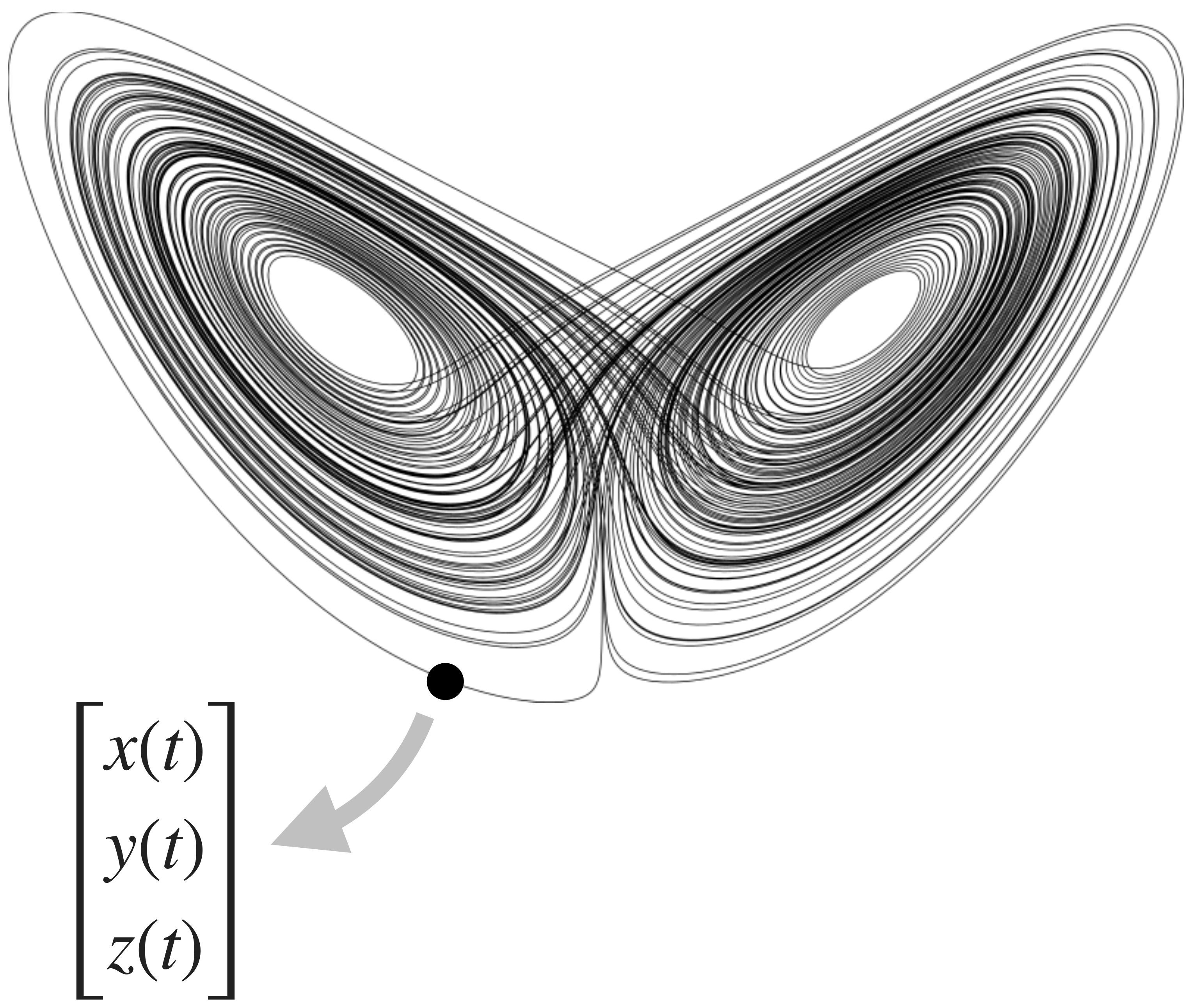
EPFL

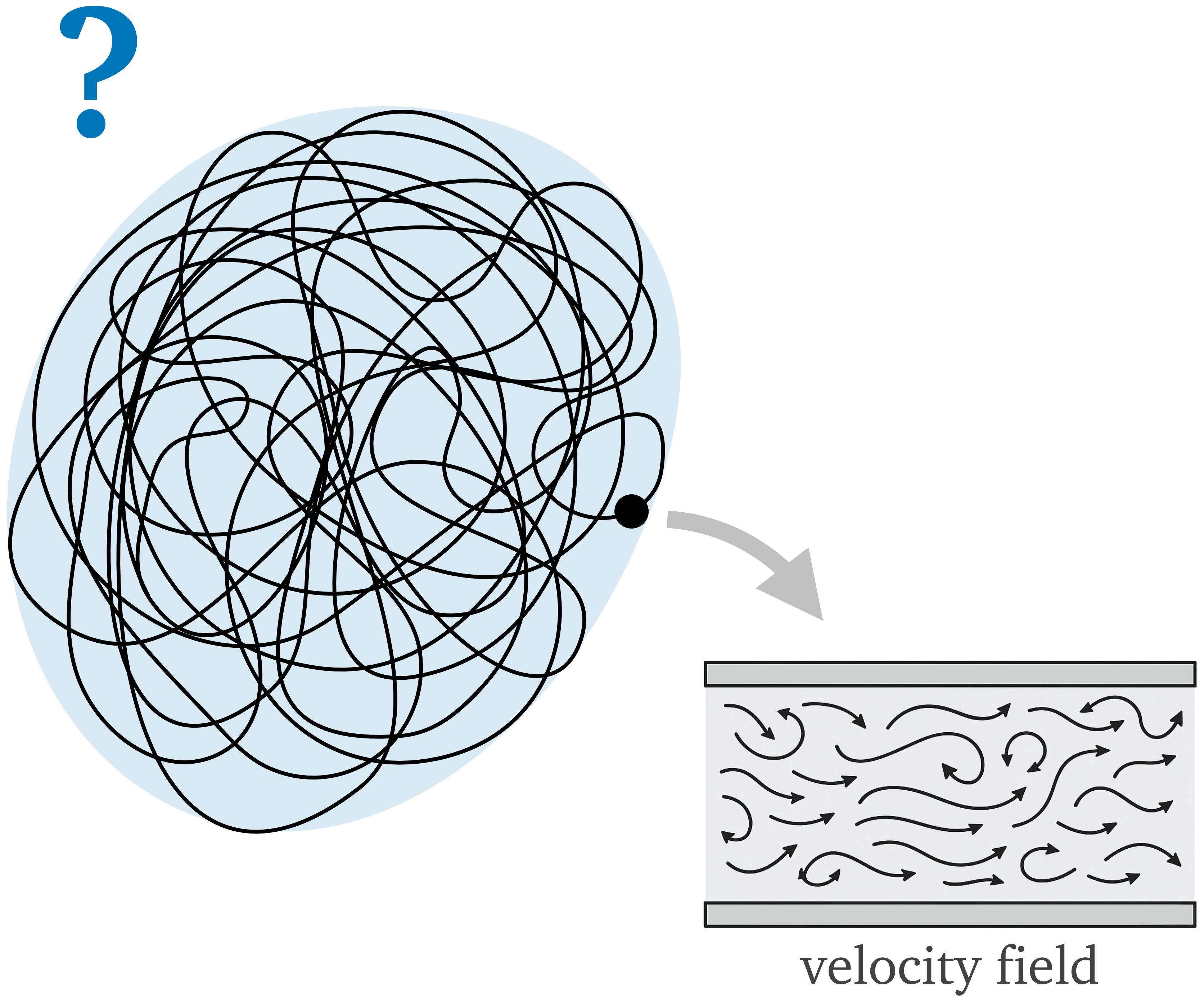
Recap

- *Chaotic attractor*

Lorenz

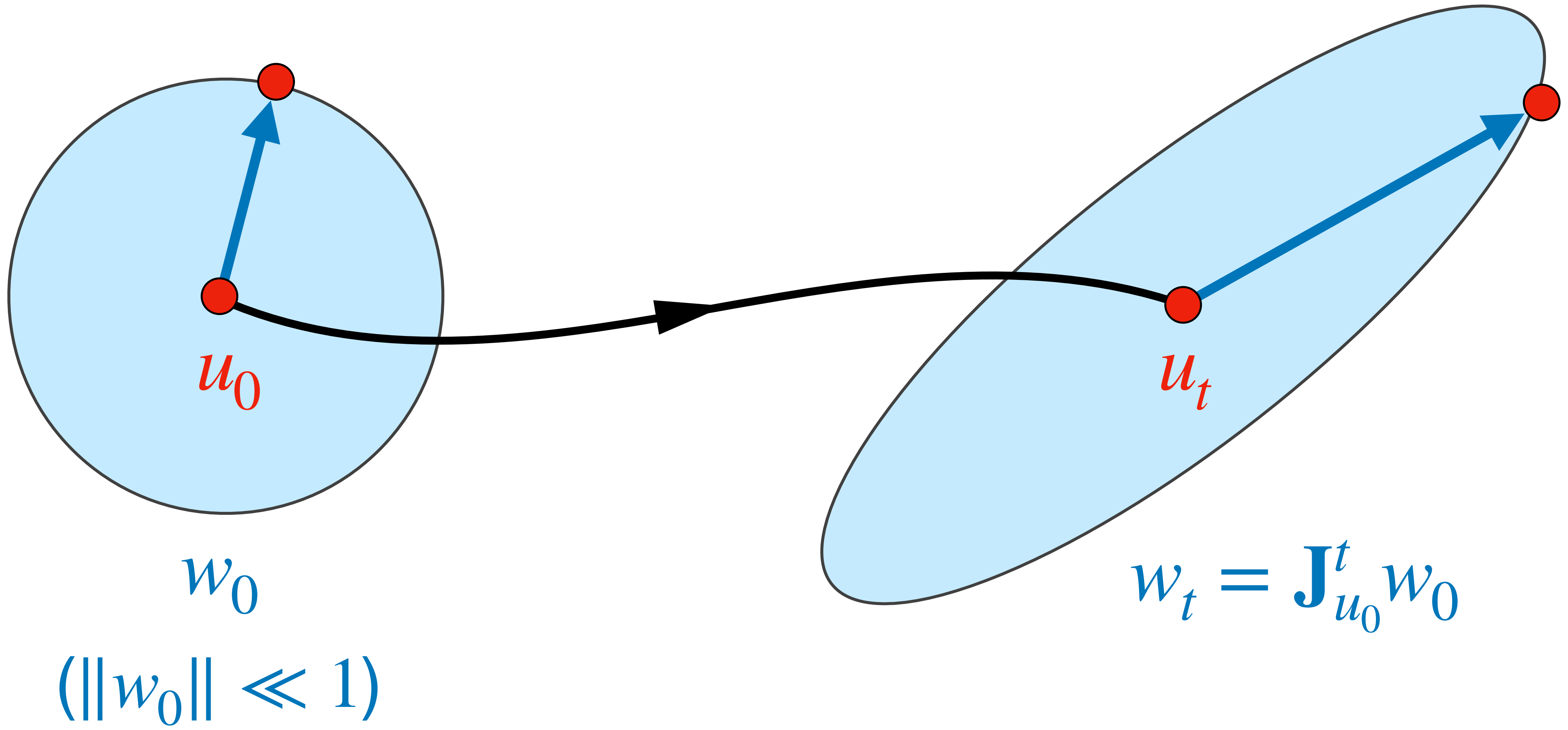


Navier—Stokes equations



Recap

- ***Jacobian*** $$\mathbf{J}^t_{u_0} := \frac{\mathrm{d}u_t}{\mathrm{d}u_0}$$
  - maps infinitesimal *neighborhoods* under the evolution.



Recap

- ***Lyapunov exponents***

  $\chi_1 \geq \chi_2 \geq \ldots \geq \chi_n$

$$\sum_{i=1}^{p} \chi_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\mathrm{vol}(t)}{\mathrm{vol}(0)}$$

$\mathrm{vol}(0)$: initial 'volume' in any p-dim. subspace

- Necessary condition for chaos $\chi_1 > 0$

- ***Lyapunov time*** $t_L^* := 1/\chi_1$
  - smaller t_L^* $\Rightarrow$ faster perturbation growth $\Rightarrow$ less predictable

- Zero Lyapunov exponents: corresponding to
  - perturbation along the orbit
  - each continuous translational symmetry $\Rightarrow$ unless prevented by a discrete symmetry.

Quiz 1

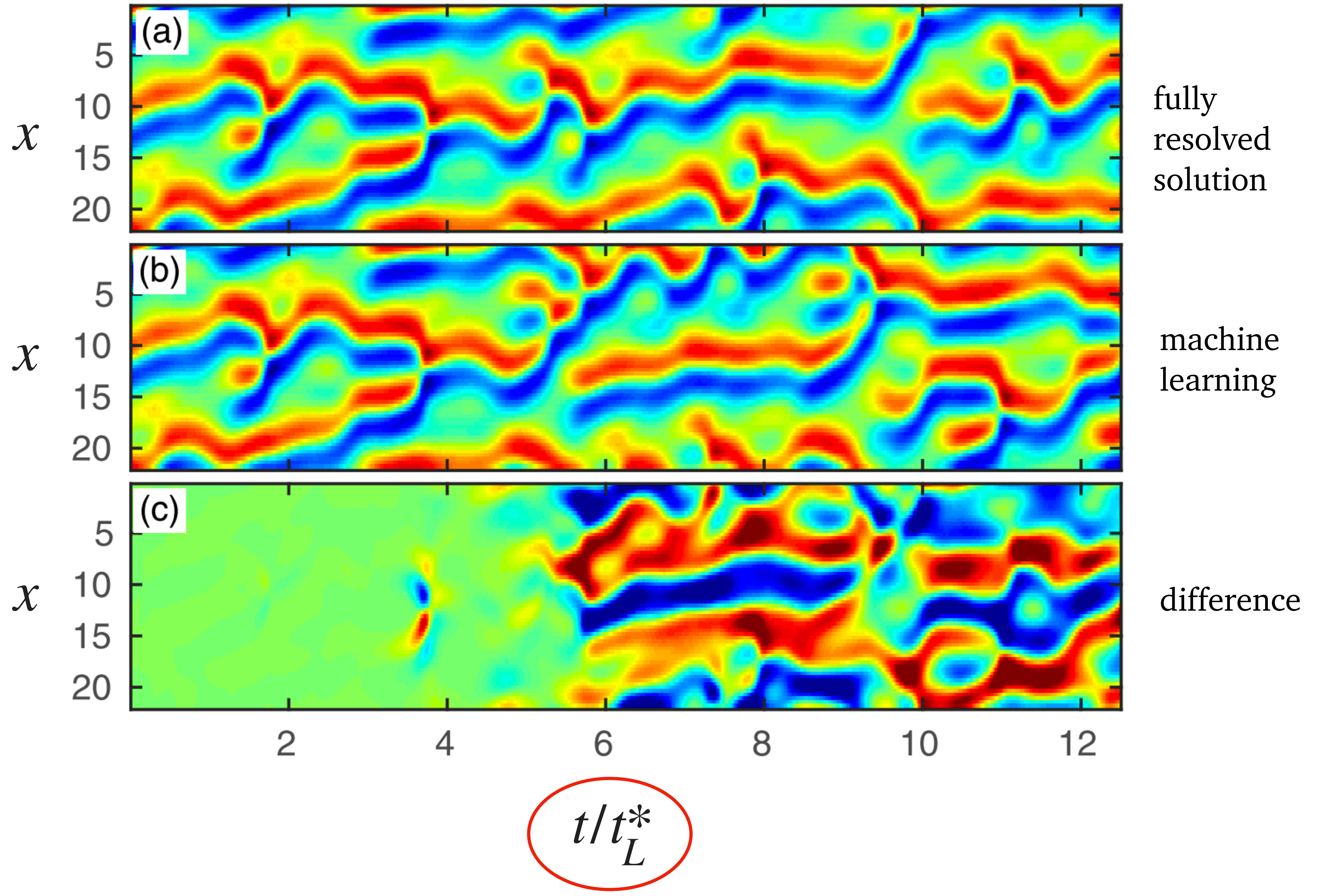
- *"An AI model replicates chaotic trajectories for a long time at a lower computational cost than fully resolved simulations."*

how many Lyapunov times?

Any red flags?

a) AI model
b) chaotic dynamics
c) long time
d) lower computational cost

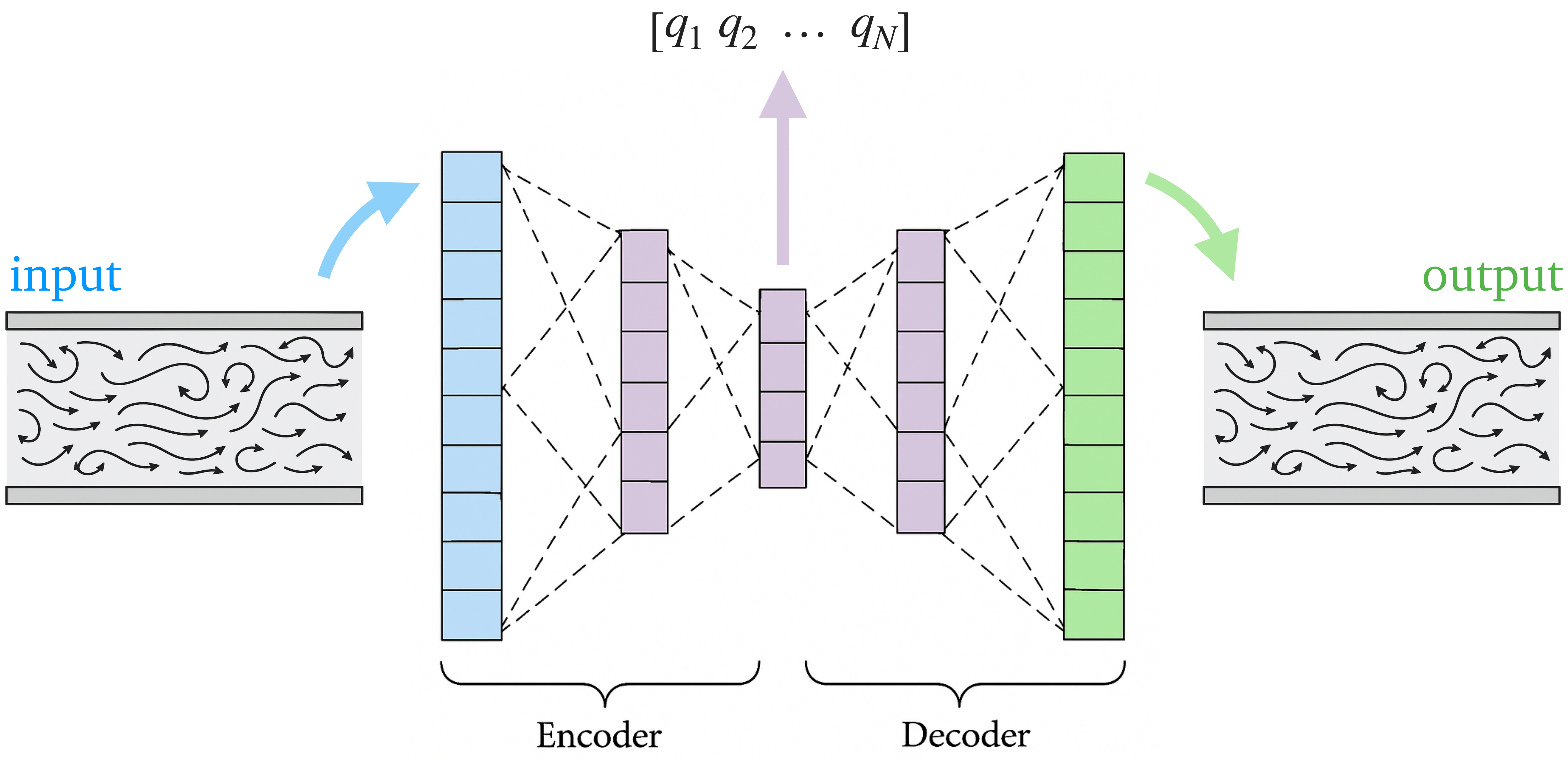
Pathak, Hunt, Girvan, Lu, Ott, *PRL* **120** (2): 024102 (2018)



Quiz 2

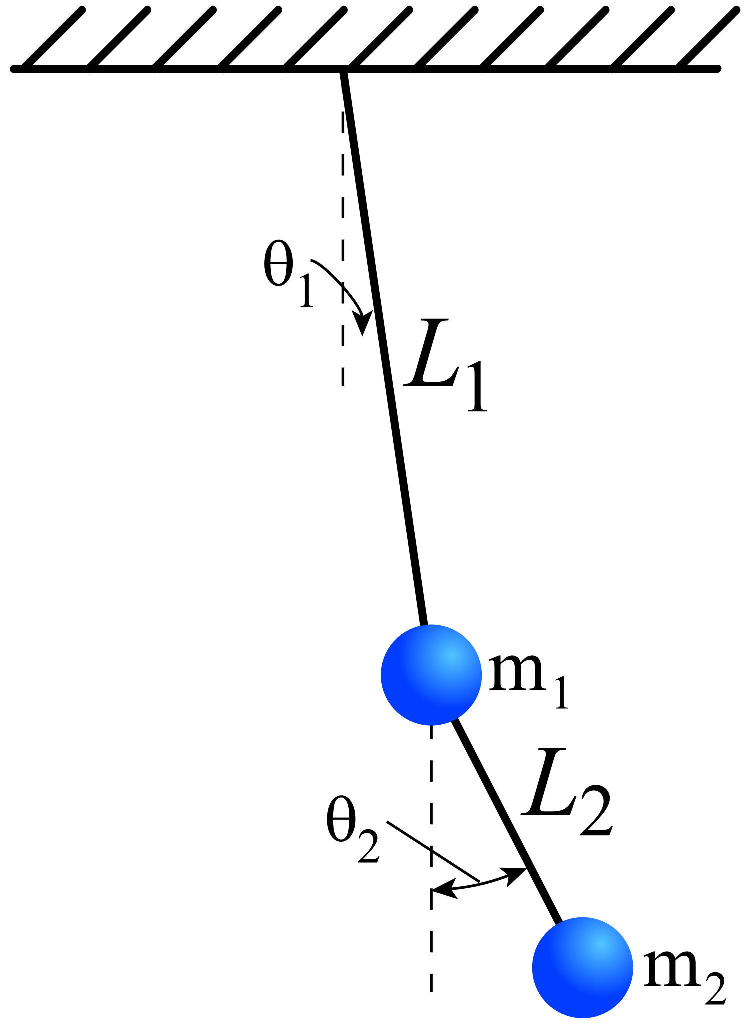
- *"A neural network is developed to learn some chaotic dynamics. The chaotic attractor is known to have a fractal dimension between j and $j+1$."*

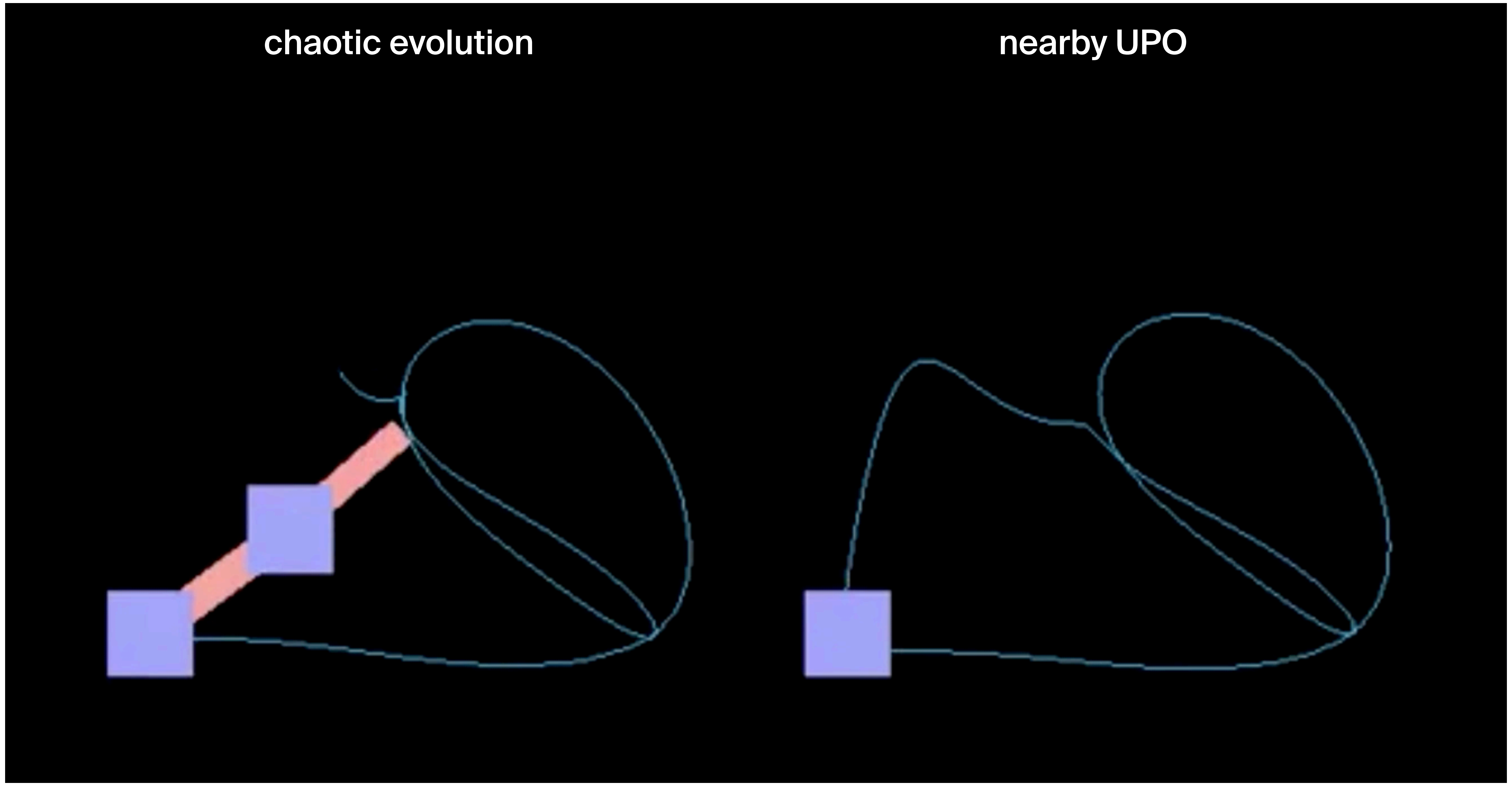
  What is a good starting point for choosing the autoencoder's latent dimension N?



Chaotic trajectories shadow unstable periodic orbits

- Example: double pendulum

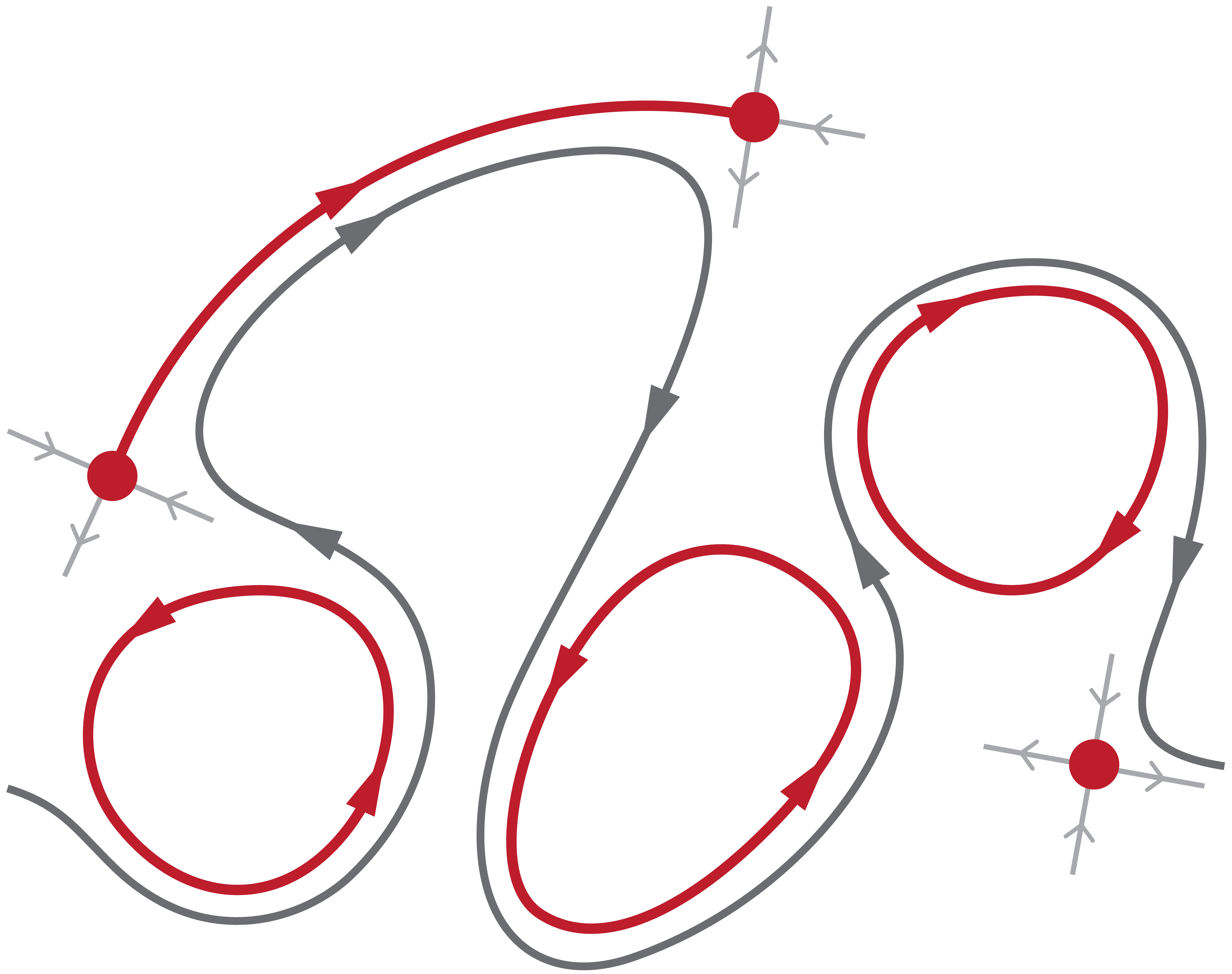




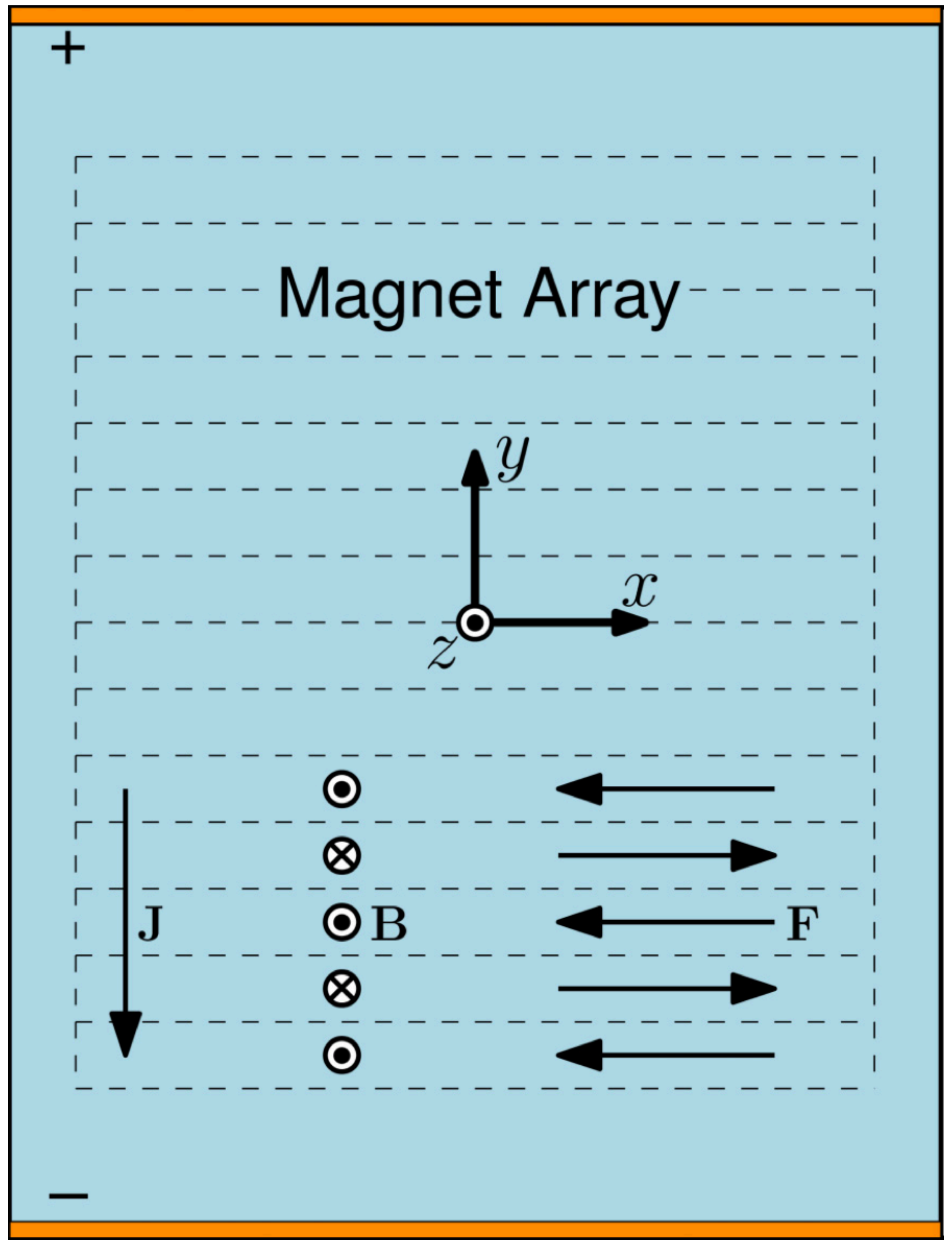
Double Double-Pendulum Android app. (by Ashley Willis)

The significance of unstable invariant solutions

- Individually
- Collectively



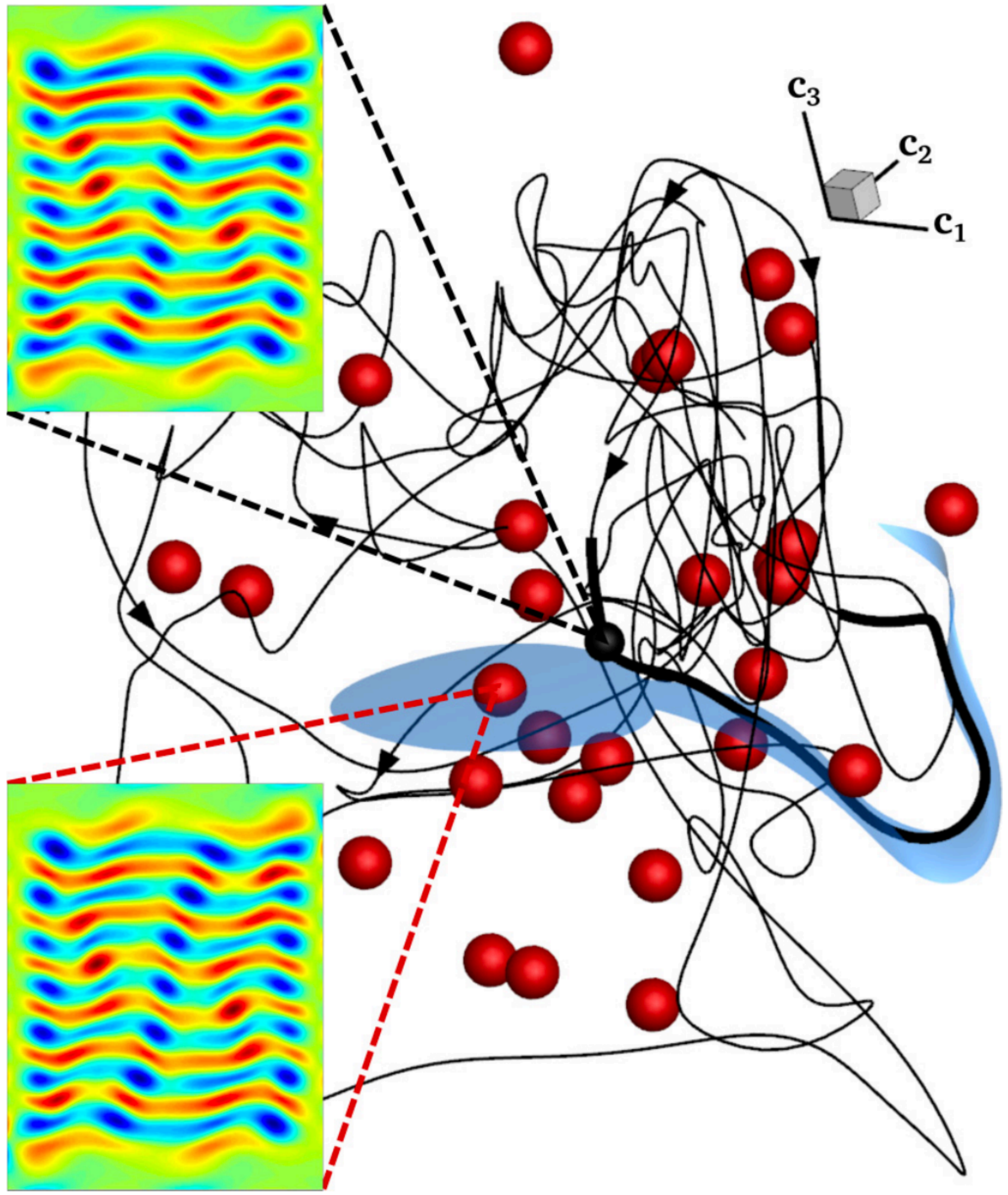
Example: 2D Kolmogorov flow



J: electric current

B: magnetic field

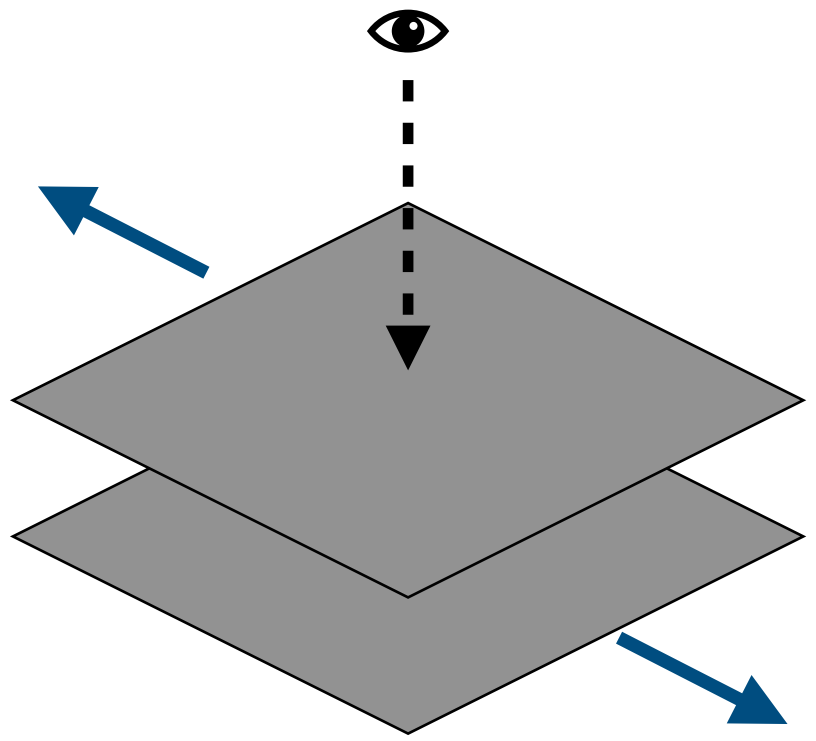
$\mathbf{F} = \mathbf{J} \times \mathbf{B}$: electromagnetic body force

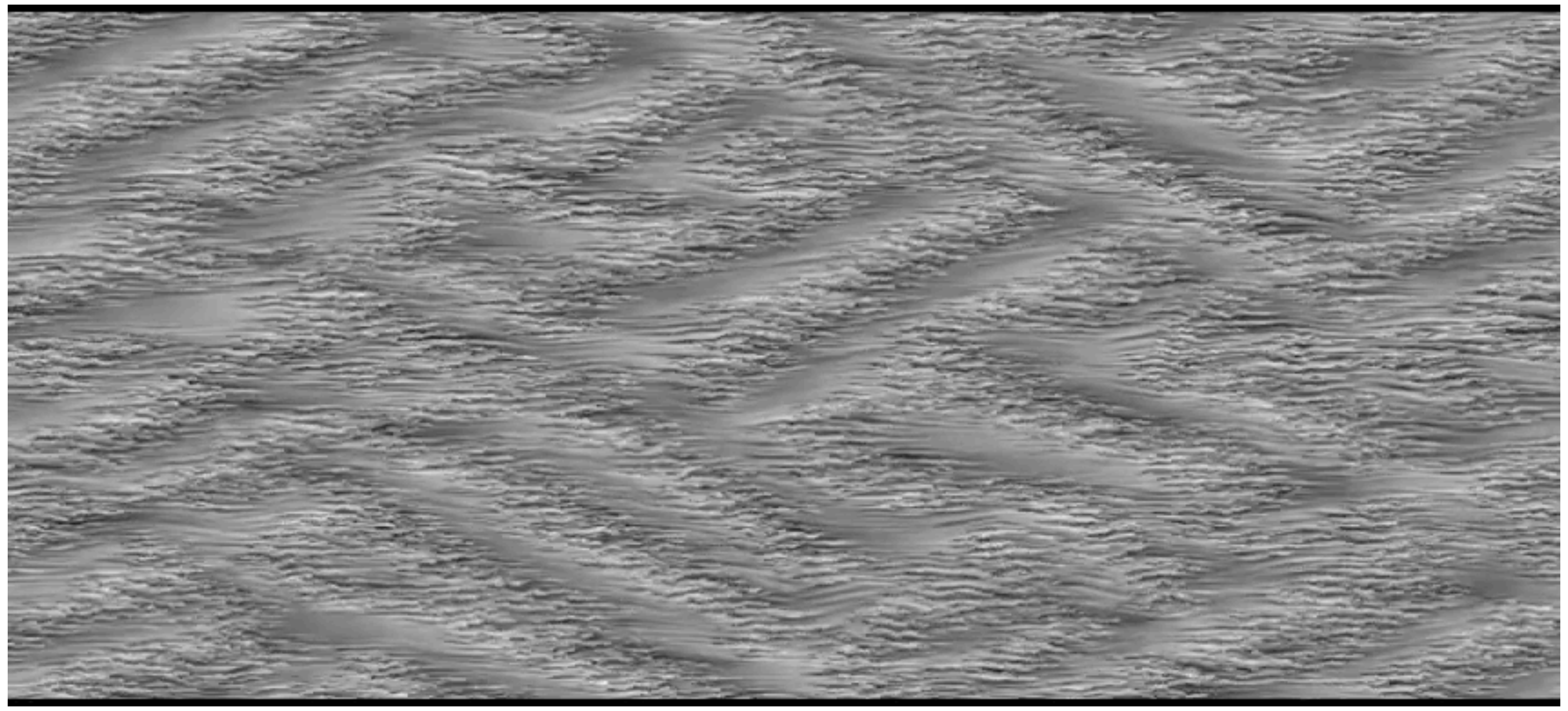


contour plots: vorticity $(\nabla \times \mathbf{u}) \cdot \hat{z}$

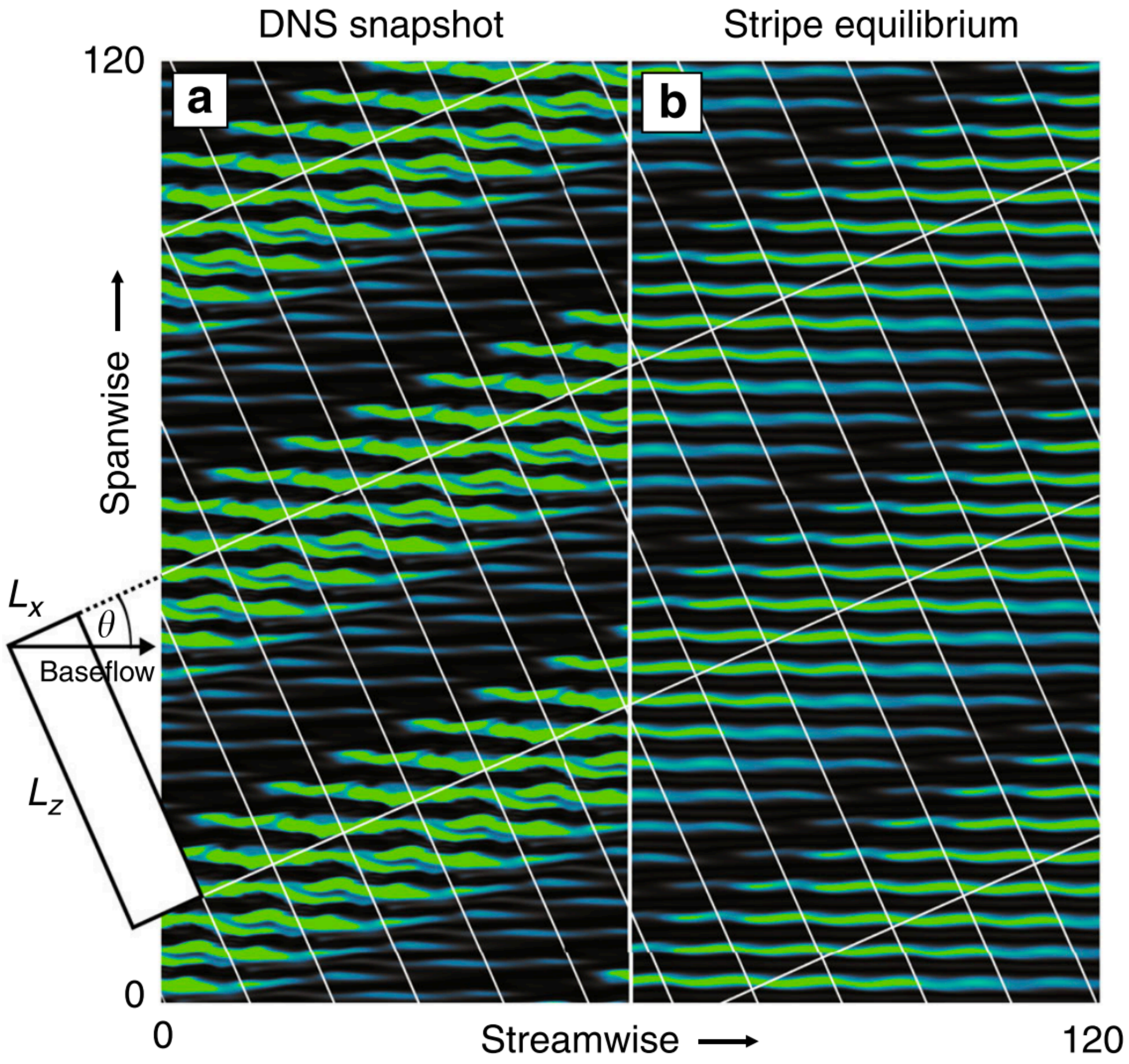
Suri, Tithof, Grigoriev, Schatz, *PRE* **98**: 023105 (2018)

Example: Plane Couette flow



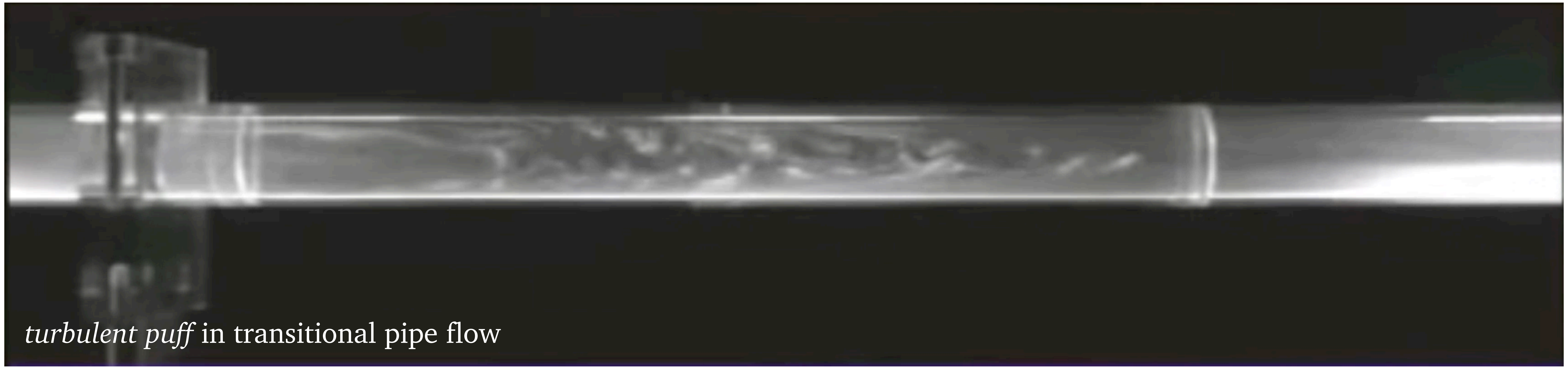


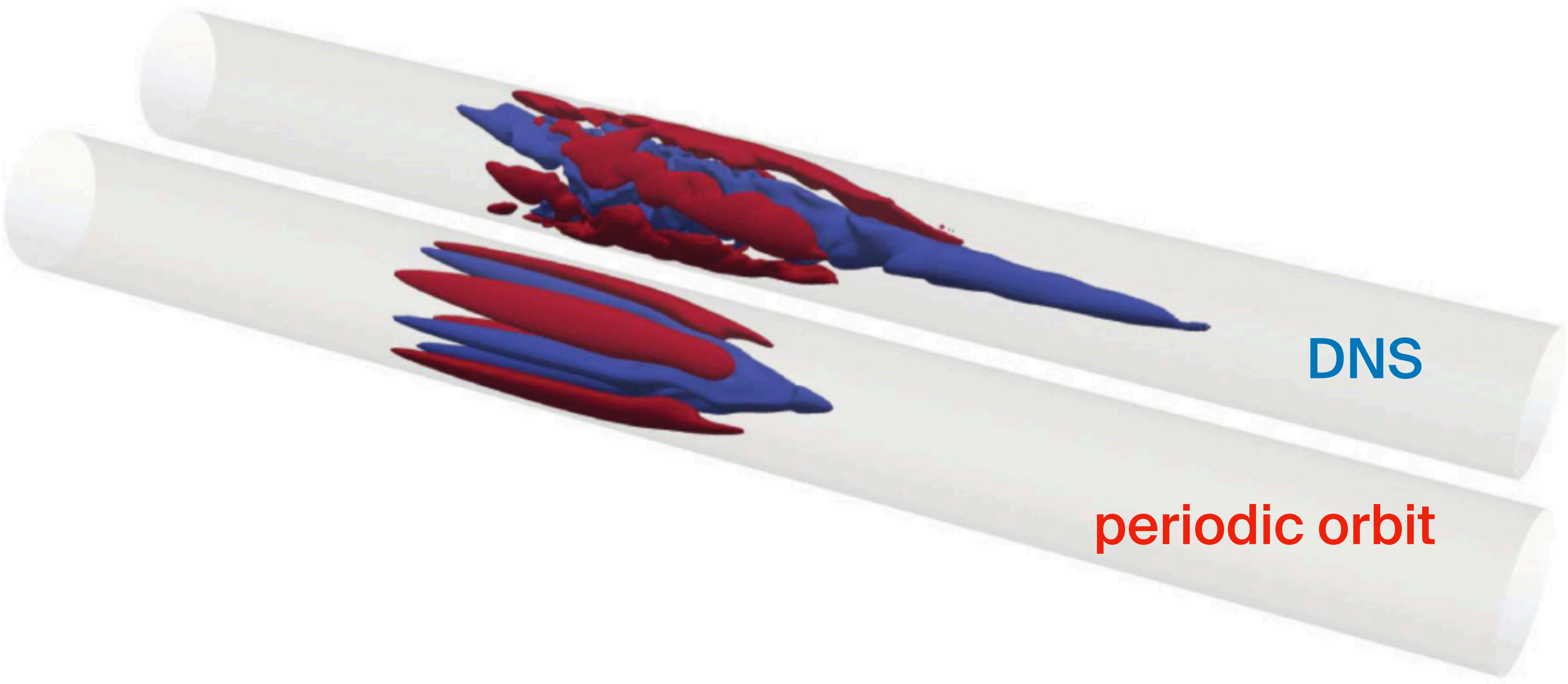
Laminar—turbulent patterns in plane Couette flow $Re = 350$

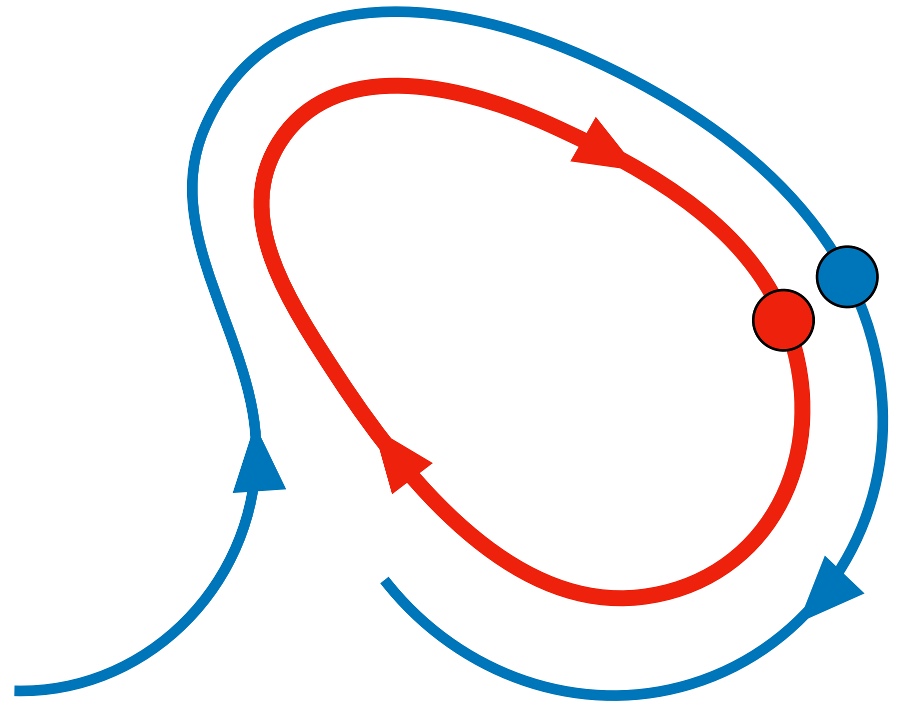


Reetz, Kreilos, Schneider, *Nature Com.* **10**: 2277 (2019)

Example: Pipe flow

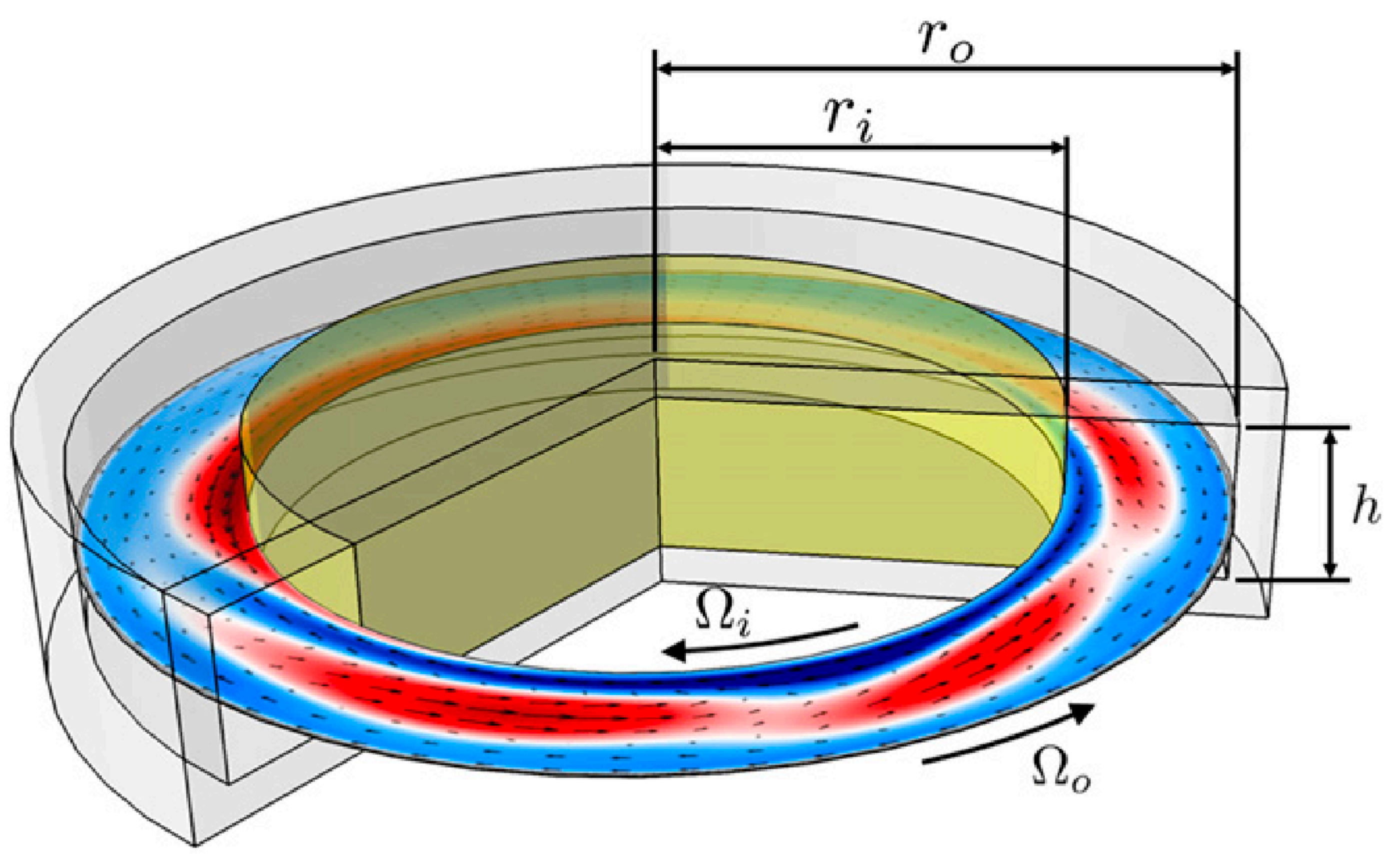


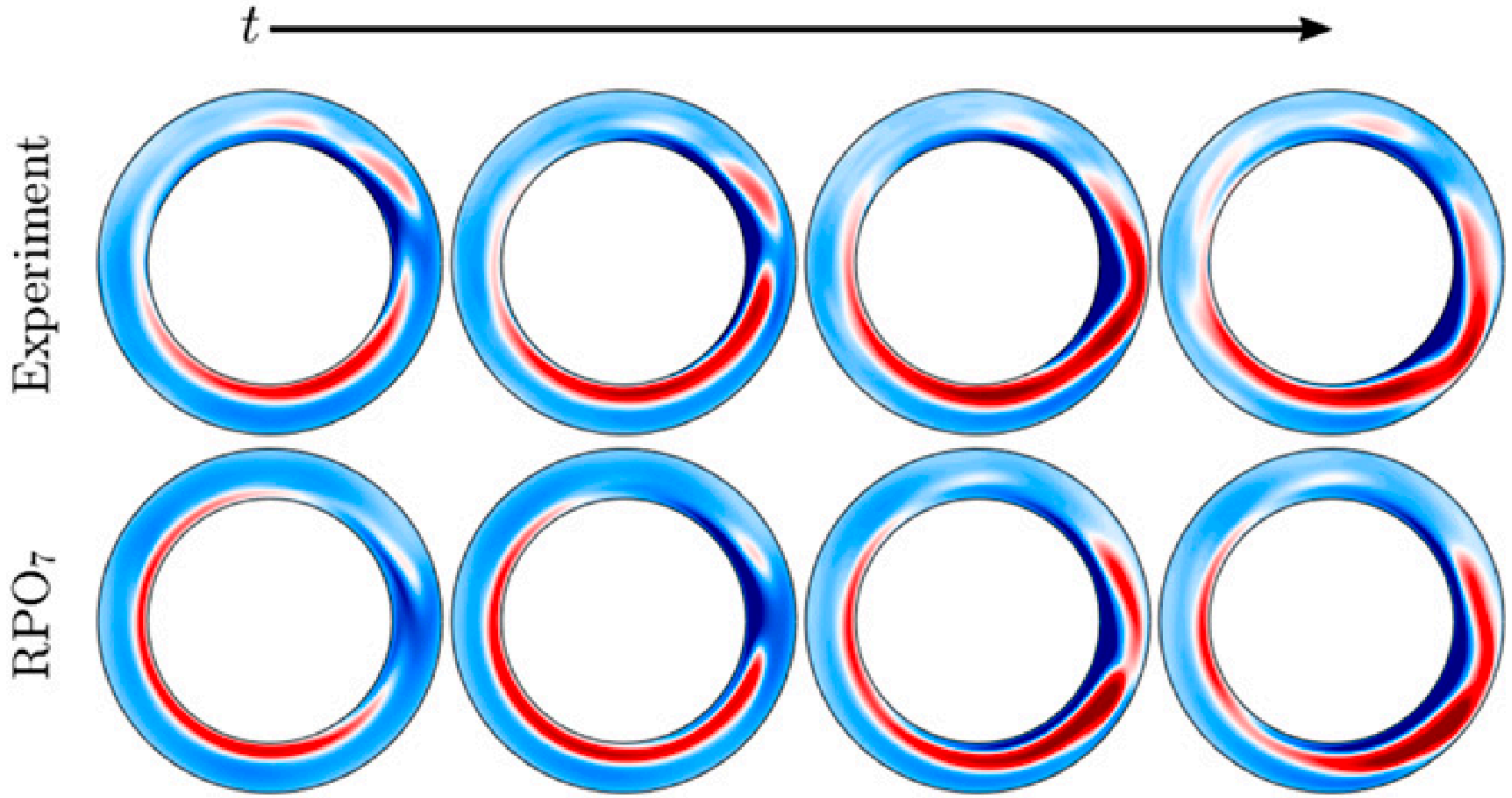




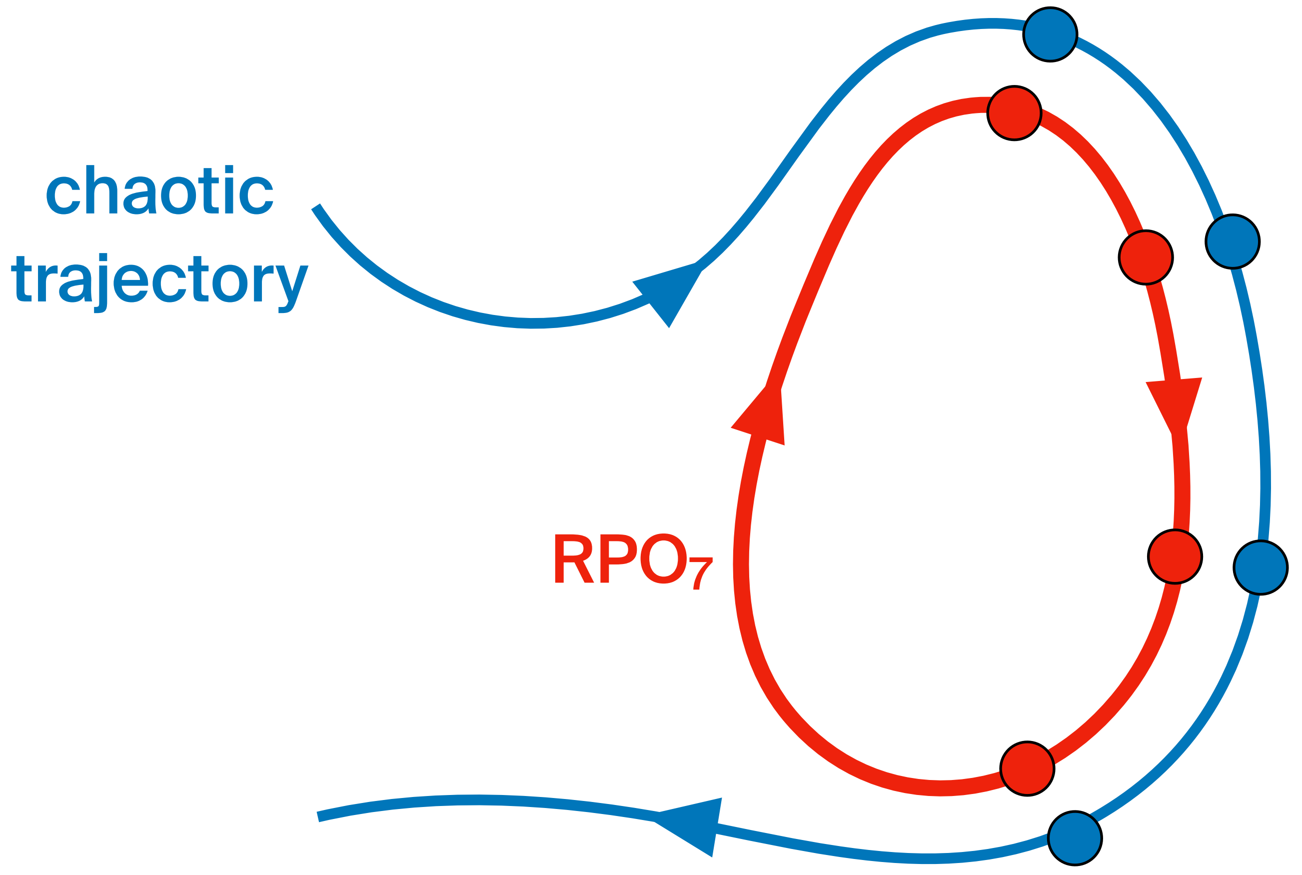
Avila, Mellibovsky, Roland, Hof, *PRL* **110** (22): 224502 (2013)

Example: Taylor—Couette flow



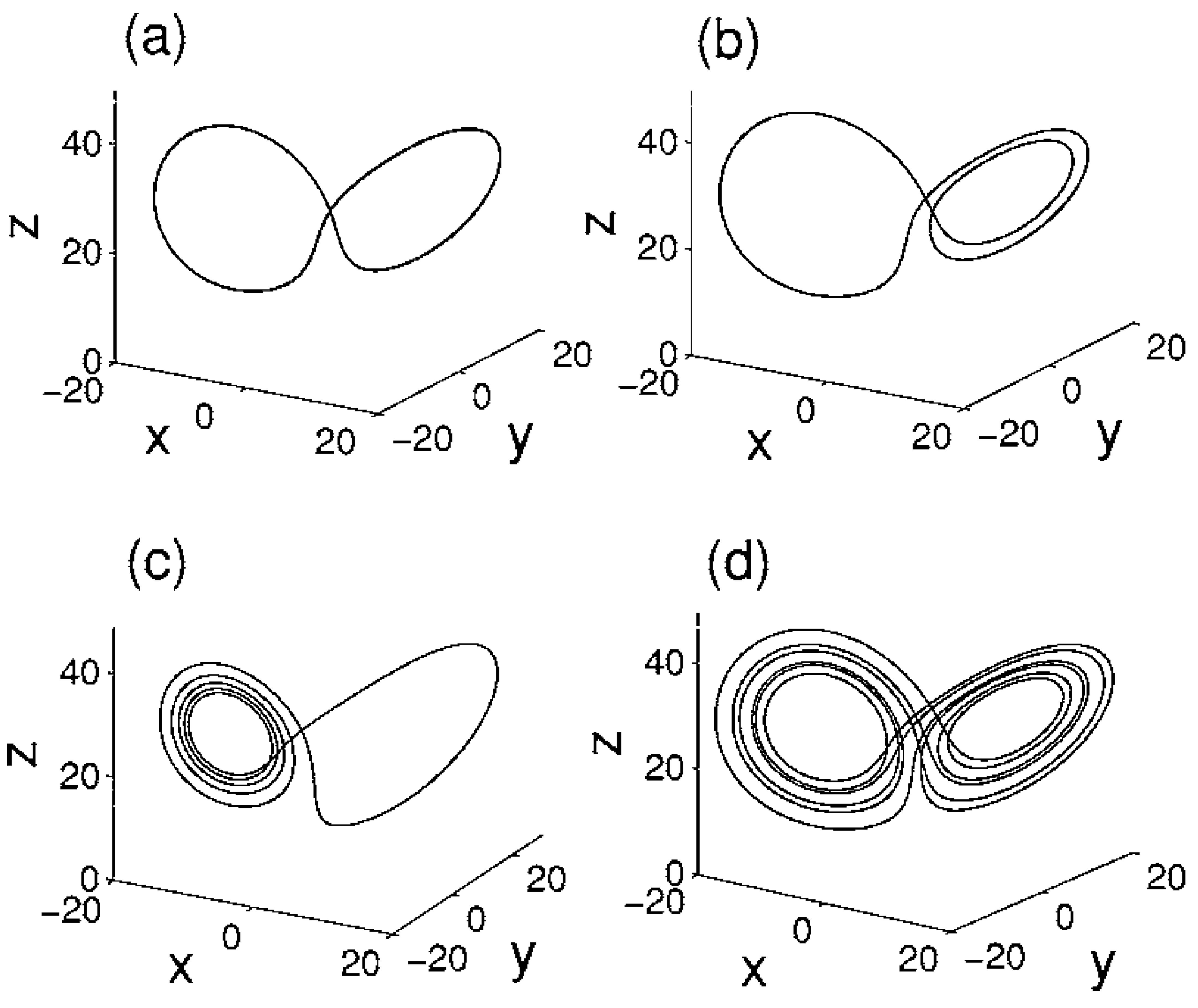


contour plots: deviation of the azimuthal velocity component u_θ from the mean



Crowley, Pughe-Sanford, Toler, Krygier, Grigoriev, Schatz, *Proc. Natl. Acad. Sci. U.S.A.* **119** (34): 1—7 (2022)

The significance of the ensemble of UPOs (Periodic Orbit Theory)



Zhao, Lai, Shih, *PRE* **72**: 036212 (2005)

the union of **all UPOs**
or
an **infinite-time trajectory**

$$\Gamma = \sum_{i=1}^{\infty} w_i \Gamma_i$$

any statistical quantity of the flow

a *single set of weights*

the same statistical quantity for the i-th UPO

- Example: mean velocity

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{u}(t)\,\mathrm{d}t = \sum_{i=1}^{\infty} w_i \left(\frac{1}{T_i}\int_0^{T_i} \mathbf{u}_i(t)\,\mathrm{d}t \right)$$

The significance of the ensemble of UPOs (Periodic Orbit Theory)

- Periodic orbit theory: the determination of w_i based on the stability properties of the UPOs
  - different from fitting!
  - hierarchy of UPOs $\Rightarrow$ truncate the infinite sum

weak driving
(small Re)

strong driving
(high Re)

spatio-temporal chaos

periodic orbit theory: $\Gamma \approx \sum_{i=1}^{n} w_i \Gamma_i$

homogeneous, isotropic, infinite-Re turbulence

K41 theory: scaling relations