Exercise Set 7: Frozen Flow and the Turbulent Cascade

1 Taylor's Frozen Flow Hypothesis

Turbulence theory makes predictions about the scaling of structure functions in space. To confirm these predictions in a experiment, measuring the flow field at many different locations simultaneously is difficult (if not impossible). It is much easier to measure a time series of one or more velocity components with a single sensor at a fixed location. *Taylor's frozen flow hypothesis* allows to transform the time signal of the velocity at a single location to a signal of velocity values at different locations.

Consider a velocity component $u(x,t)$ of a turbulent flow that streams past a sensor at a large mean velocity U. Using the following steps, show that if the velocity fluctuations are much smaller than the mean velocity,

$$\frac{\sqrt{\langle \hat{u}^2 \rangle}}{U} \ll 1, \tag{1}$$

(where $\hat{u}(x,t) \equiv u(x,t) - U$), then a velocity difference in time can be written as a velocity difference in space as

$$u(x,t+\tau) - u(x,t) \;\approx\; u(x-l,t) - u(x,t), \tag{2}$$

with $l \equiv U\tau$ and $\tau \ll 1$.

a) Use the Galilei transform $(t,x,u) \to (t, x-Ut, u-U)$ to write $u(x,t+\tau) - u(x,t)$ in a co-moving reference frame, i.e. in terms of $\hat{u}$.

b) Set $U\tau = l$. Since U is large and τ is small, l is neither large nor small.

c) Transform the system back into the lab frame, assuming that $\hat{u}(r,t+\tau) \approx \hat{u}(r,t)$ for any r. (Do you see why this might be a good approximation?)

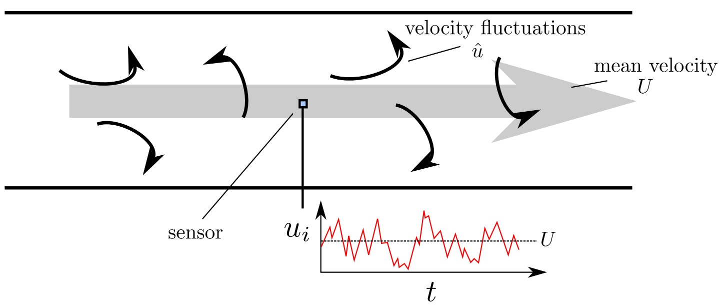

Figure 1: A sensor in a wind tunnel measures a time series of some velocity component.

2 Local Interactions and the Turbulent Cascade

In the inertial range, energy transport is *local*. That means that eddies of size l exchange energy mostly with eddies of similar size $l' \approx l$. This local coupling (in k-space) is a key element of K41 turbulence theory and underlies the Richardson cascade picture. In class, we demonstrate the local transport by analyzing the explicit expression for the energy flux Π_K in homogeneous isotropic turbulence. Here we aim at a more intuitive understanding in terms of eddies in real physical space.

Argue why an eddy of size l is neither receiving energy from much larger ($l' \gg l$) nor much smaller ($l' \ll l$) eddies. To do this, assume the flow to be formed by a superposition of eddies of various size. Energy transport from one eddy to another is associated with the flow of one eddy deforming the other.

Influence of Large Eddies on Small Ones

Consider the advection of a small eddy by a large one (Figure 2a). The typical velocity difference of an eddie of size l in the inertial range scales like $v_l \sim \epsilon^{1/3} l^{1/3}$ (where ϵ is the global dissipation).

a) What does that scaling imply for the shear (velocity difference per length)?

b) Is the shear of large or small eddies larger? (Hint: Does ϵ depend on l in the inertial range?)

c) The deformation of a small eddy scales with its size l times the shear induced by the large eddy. What happens with this interaction when $l' \gg l$?

This is why much larger eddies do not transfer much energy to smaller eddies.

Influence of Small on Large Eddies

Consider a large eddy of size l advecting small eddies of size l' embedded in it (Figure 2c). The large eddie will be deformed by the collective effect of the small eddies, i.e. the mean shear.

a) What is the collective effect? (Hint: Shear is associated with a shear axis. The flow is assumed to be statistically isotropic.)

This is why much smaller eddies do not transfer much energy to larger eddies.

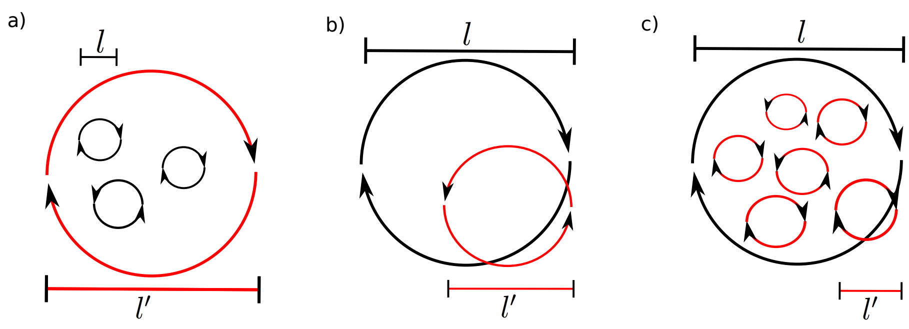

Figure 2: Influence of eddies of size l' (red) on eddies of size l (black). a) Influence of large eddies on small ones. b) Interaction between similar-sized eddies. c) Influence of many small eddies on a large one.