ME-467: Turbulence 04 April 2025
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Exercise Set 6: Probability and Structure Functions

1 Sampling a Probability Distribution

In measuring an observable from a dynamical system, the structure of a time series or sample series is often hard to describe. In the Matlab script `moments.m`, the function `genrandom(n,prm)` creates a `n` $\times$ 1-vector of samples from some (usually unknown) distribution. `prm` ($\in [0 : 1]$) is a real-valued parameter that changes the distribution of the output variables.

a) For `n`=1000, look at the sample of the measurement for x (see Figure 1 of `moments.m`).
   - Describe the change in the graph between `prm`=0 and `prm`=0.5.

b) Figure 2 shows the probability density function (PDF) of the sample (blue), and a normal distribution of the same mean and standard deviation (green). The same data is shown on a linear scale on the left, and in a semilogarithmic plot on the right.
   - Compare the PDF between different values of `prm` ($\in [0 : 1]$): What is the qualitative effect of the parameter?

c) Figure 3 shows how the values of mean, variance, skewness and flatness converge to their final value as you increase the sample size.
   - Set `n`=10^5 or larger, `prm`=$0, 0.5, 1$, and estimate the final values.
   - At `prm`=0.5, the distribution is called *skewed*. What does that mean?
   - Can you see this already from looking at the probability distribution in Fig. 2?

2 Structure Functions

Consider a random variable $v(t)$ (e.g. the component of a velocity), which is centered ($\langle v \rangle = 0$) and stationary. The energy spectrum $E(f)$ has the properties

$$\frac{1}{2}\langle v^2 \rangle = \int_0^\infty E(f)\mathrm{d}f \qquad \text{(from Parseval's theorem)} \qquad (1)$$

$$\frac{1}{2}\langle \left(\tfrac{\mathrm{d}v}{\mathrm{d}t}\right)^2 \rangle = \int_0^\infty f^2 E(f)\mathrm{d}f \qquad \text{(from Parseval's theorem)} \qquad (2)$$

$$E(f) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ifs}\Gamma(s)\mathrm{d}s \qquad \text{(from Wiener-Khinchin Theorem)} \qquad (3)$$

with the temporal auto-correlation function $\Gamma(s) \equiv \langle v(t+s)v(t) \rangle$.

a) E can be defined for negative arguments as $E(-f) \equiv E(f)$. Show that this definition is consistent with equation (3).

b) Show that the second-order structure function

$$S_2(s) \equiv \langle [v(t+s) - v(t)]^2 \rangle \qquad (4)$$

can be expressed in terms of E as

$$S_2(s) = 2\int_{-\infty}^\infty (1 - e^{ifs})E(f)\mathrm{d}f. \qquad (5)$$

Use the relation $\int_{-\infty}^\infty e^{ifs}\mathrm{d}f = 2\pi\delta(s)$.

c) Show that the second-order structure function in a field v with an energy spectrum $E(f) = C|f|^{-n}$ has the form

$$S_2(s) = C \cdot A_n \cdot |s|^{n-1}, \qquad (6)$$

with a constant $A_n \equiv 2\int_{-\infty}^\infty (1 - e^{ix})|x|^{-n}\mathrm{d}x$.