ME-467: Turbulence 04 April 2025
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Exercise Set 5: Probability Distributions – Solutions

Let the random variable X be distributed according to the probability density function (PDF)

$$\rho(x) \equiv \frac{2}{\pi} \frac{1}{(1+x^2)^2} \tag{1}$$

for $x \in \mathbb{R}$. The cumulative probability of X is

$$R(x) = \frac{1}{\pi} \left[\frac{x}{1+x^2} + \text{atan}(x) + \frac{\pi}{2} \right]. \tag{2}$$

1 Moments of a Distribution

a) Show that $\rho(x)$ is a PDF and its cumulative probability is $R(x)$.

b) Calculate the mean and variance of X.

c) Plot $\rho(x)$ and compare it to a normal distribution of equal mean and variance.

d) If possible, find the skewness and flatness of X.

e) Describe (in words) the higher order moments of X.

Solution:

a) $\rho(x)$ is a PDF because $\rho(x) \geq 0 \ \forall x$ and it is normalized, i.e. $\int_{-\infty}^{\infty} \rho(x)\mathrm{d}x = 1$. To check this, we can calculate the integral

$$\begin{aligned}
\int \frac{dx}{(1+x^2)^2} &= \int \frac{du}{(1+\tan^2 u)^2 \cos^2 u} \quad \text{via} \quad x = \tan u \\
&= \int \cos^2 u \, du \\
&= \frac{1}{2} \int (\cos 2u + 1) \, du \\
&= \frac{1}{4} \sin 2u + \frac{1}{2} u + \text{const.} \\
&= \frac{1}{2} \sin u \cos u + \frac{1}{2} \arctan x + \text{const.} \\
&= \frac{1}{2} \frac{x}{\sqrt{1+x^2}} \frac{1}{\sqrt{1+x^2}} + \frac{1}{2} \arctan x + \text{const.} \\
\implies \int \frac{dx}{(1+x^2)^2} &= \frac{1}{2} \frac{x}{1+x^2} + \frac{1}{2} \arctan x + \text{const.}
\end{aligned}$$

Thus

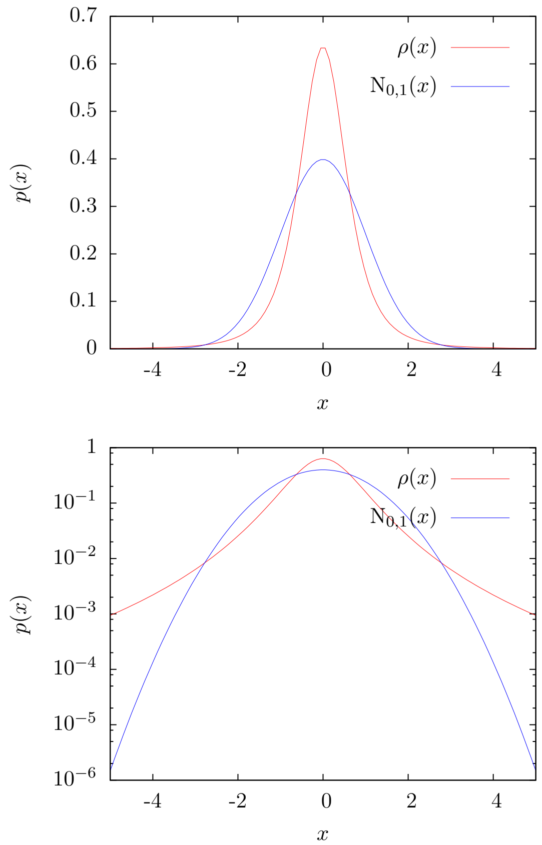
$$\int_{-\infty}^{+\infty} \rho(x)\, dx = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} = \frac{2}{\pi}\frac{1}{2}\left[\frac{x}{1+x^2} + \arctan x\right]_{-\infty}^{+\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1$$

Finally, since $\frac{dR}{dx} = \rho(x)$ and $\lim_{x\to+\infty} R(x) = 1$, $R(x)$ is the cumulative distribution function of $\rho(x)$.

b) The mean is $\langle X\rangle = \int_{\mathbb{R}} x\rho(x)dx = 0$ as x is odd along $x = 0$, $\rho(x)$ is even along $x = 0$, and the integration interval is symmetric about $x = 0$.

The variance is $\langle X^2\rangle = \int_{\mathbb{R}} x^2\rho(x)dx = \frac{2}{\pi}\int_{-\infty}^{+\infty} \frac{x^2}{(1+x^2)^2}dx = 0 + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx = 1$, where we used integration by parts.

c) On a linear scale (top figure), it looks like distribution $\rho(x)$ is much more peaked and quickly decays to zero for large $|x|$. In a semi-logarithmic plot (bottom figure), it becomes clear that the extreme values are actually much more likely for $\rho(x)$, because the exponential decay of the Gaussian is much faster than a polynomial.



d) The skewness is given by $S = \frac{\langle X^3\rangle}{\langle X^2\rangle^{3/2}} = \langle X^3\rangle = \frac{2}{\pi}\int_{-\infty}^{+\infty} x^3/(1+x^2)^2dx$. While it is tempting to use a symmetry argument again here, note that as $|x| \to +\infty$, the integrand behaves like $x^3/(1+x^2)^2 \sim \mathcal{O}(x^{-1})$, and so the integral is undefined. For the same reason, the *Cauchy distribution* does not have a well-defined mean.

The flatness $F = \frac{\langle X^4\rangle}{\langle X^2\rangle^2} = \langle X^4\rangle$ diverges to $+\infty$.

e) None of the odd moments $\langle X^{2n-1}\rangle$ for $n = 2, 3, \ldots$ are defined. The even moments $\langle X^{2n}\rangle$ for $n = 2, 3, \ldots$ diverge to $+\infty$.

2 Transforming Probability Distributions

Consider the random variable Y on $y \in [0, \infty)$, defined by

$$Y = \exp(X), \tag{3}$$

with X distributed according to some PDF $\rho(x)$ with cumulative probability $R(x)$.

a) Find the cumulative probability $F_Y(y) = \mathbb{P}(Y < y)$.

b) Find the PDF $f_Y(y)$ of Y.

c) Determine the cumulative probability and PDF for $Z = \exp(-X)$.

Solution:

a)

$$\begin{aligned} F_Y(y) &\equiv \mathbb{P}(Y < y) \\ &= \mathbb{P}(\exp(X) < y) \\ &= \mathbb{P}(X < \ln(y)) && \text{since exp is monotonic} \\ &= R(\ln(y)) && \text{with } R(x) \text{ the cumulative probability of X.} \end{aligned}$$

b)

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) && \text{by definition} \\ &= \left.\frac{dR}{dx}\right|_{x=\ln(y)} \cdot \frac{d(\ln(y))}{dy} && \text{by chain rule} \\ &= \rho(\ln(y)) \cdot \frac{1}{y} && \text{by definition of the cumulative probability } R \end{aligned}$$

c) This follows the same argument, except that exp decreases monotonically, which switches the $<$ (smaller than) to $>$ (larger than):

$$\begin{aligned} F_Z(z) &\equiv \mathbb{P}(\exp(-X) < z) \\ &= \mathbb{P}(-X < \ln(z)) \\ &= \mathbb{P}(X > -\ln(z)) \\ &= 1 - \mathbb{P}(X < -\ln(z)) && \text{this is the important step!} \\ &= 1 - R(-\ln(z)) \end{aligned}$$

Then, we get

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= -\left.\frac{dR}{dx}\right|_{x=-\ln(z)} \cdot \frac{d(-\ln(z))}{dz} \\ &= \rho(-\ln(z)) \cdot \frac{1}{z} \end{aligned}$$